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Group theoretical approach to quantum fields in de Sitter space II. The complementary and discrete series

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Abstract: We use an algebraic approach based on representations of de Sitter group to construct covariant quantum fields in arbitrary dimensions. We study the complementary and the discrete series which correspond to light and massless fields and which lead new feature with respect to the massive principal series we previously studied (hep-th/0606119). When considering the complementary series, we make use of a non-trivial scalar product in order to get local expressions in the position representation. Based on these, we construct a family of covariant canonical fields parametrized by SU(1, 1)/U(1). Each of these correspond to the dS invariant alpha-vacua. The behavior of the modes at asymptotic times brings another difficulty as it is incompatible with the usual definition of the *in* and *out* vacua. We propose a generalized notion of these vacua which reduces to the usual conformal vacuum in the conformally massless limit. When considering the massless discrete series we find that no covariant field obeys the canonical commutation relations. To further analyze this singular case, we consider the massless limit of the complementary scalar fields we previously found. We obtain canonical fields with a deformed representation by zero modes. The zero modes have a dS invariant vacuum with singular norm. We propose a regularization by a compactification of the scalar field and a dS invariant definition of the vertex operators. The resulting two-point functions are dS invariant and have a universal logarithmic infrared divergence.

Keywords: Space-Time Symmetries, Global Symmetries, dS vacua in string theory.

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Contents

1. Introduction

Since the works of Dirac and Wigner [1], particles in flat spacetime can be considered as unitary irreducible representations (UIR) of the Poincaré group. Moreover, free quantum field operators $\Phi(x)$ can be constructed in a unique way from these UIR [2, 3]. The key property in this construction is the covariance of the field operator:

$$\Phi(x') = U(\Lambda)\,\Phi(x)\,U^{\dagger}(\Lambda), \tag{1.1}$$

where we only consider scalar fields and x' is the image of x under the Poincaré transformation Λ.

In a previous work [4], hereafter cited as I, we presented an algebraic construction of quantum field theories (QFT) on the n-dimensional de Sitter space dS_n based on the UIRs of the de Sitter isometry group $\mathrm{SO}_0(1,n)$. The UIR were first analyzed by Bargmann [5] for $n = 2$, Gelfand and Naimark for $n = 3$ [6], Thomas, Newton and Dixmier for $n = 4$ [7−9]. Generalization to all n were studied in [10−13]. Starting with an UIR, we first construct the corresponding Fock space. The vacuum is the trivial representation, the one-particle states are elements of the UIR, and the n-particle states are the symmetrized tensor product of n copies of the UIR. The dS group has an induced representation U on this Fock space. We then define a free local field operator by a linear superposition of creation and annihilation operators subject to the covariance property (1.1) with Λ now being an element of $\mathrm{SO}_0(1,n)$. The scalar representations are of three types characterized by the eigenvalue of the quadratic Casimir operator $\mathcal{C}$: the principal series has $\mathcal{C} \leq -(n-1)^2/4$, the complementary series has $-(n-1)^2/4 \leq \mathcal{C} < 0$ and finally the discrete series has $\mathcal{C} = k(k+n-1)$ with $k \in \mathbb{Z}_{\geq 0}$. The eigenvalue of $\mathcal{C}$ can be interpreted physically as the minus mass squared: $\mathcal{C} = -M^2$ (or more generally with a curvature coupling term, $\mathcal{C} = -(M^2 + \xi\,\mathcal{R})$). In I, we studied the principal series and we showed that the construction gives rise to a family of canonical fields parametrized by a $\mathrm{SU}(1,1)/\mathrm{U}(1)$ moduli space. In the usual field theoretical treatment, this moduli space corresponds to that of alpha-vacua [14]. In our approach, the moduli space stems from a first order differential equation expressing the invariance of the field operator under transformations leaving fixed a given point in dS space. This first order differential equation turns out to be singular (on the event horizons centered about that point) thereby giving rise to two independent complex solutions. Furthermore, we showed that the Klein-Gordon equation and the canonical commutation relations are consequences of the covariance requirement.

The aim of the present work is to extend this approach to the other two series. Concerning the complementary series, the main difference with respect to the principal one is that the realization of the representation with functions on the sphere is no more unitary with respect to the standard $\mathcal{L}^2$ scalar product. This realization was very convenient in I because the generators are local differential operators and the finite transformations easy to determine. The unitarity of the complementary series can be recovered by the use of a different scalar product which we first determine. The construction then follows the same general line as before leading once again to a family of free field operators labeled by the moduli space $\mathrm{SU}(1,1)/\mathrm{U}(1)$. An important physical difference with respect to the principal series exists however, it concerns the different behavior of the field operators in the asymptotic past and future. This forbids the usual definition of the *in* and *out* Mottola-Schwinger vacua [15]. We propose a new definition of asymptotic vacua by appropriately factorizing the field operator and choosing the time coordinate. Our construction generalizes the conformal *in* and *out* vacua valid for the conformally coupled scalar fields with $\mathcal{C} = -n(n-2)/4$.

As of the discrete series, we consider only the physically interesting case of the first discrete series with $\mathcal{C} = 0$, i.e. the massless case. Other discrete series correspond to the tachyonic fields which are physically less relevant (see [16] for an example). The massless scalar case is peculiar in the usual field theoretic approach since the dS invariant two-point function diverges in the limit $M \to 0$ [17]. This was interpreted as there is no dS invariant vacuum in the massless case [14] and the vacuum states breaking dS group but preserving its subgroup were considered in [14, 18−20]. It was suggested that this divergence is related to the additional symmetry which the massless theory acquires: the symmetry under the constant addition on the field operator. In order to implement this symmetry appropriately, the BRST quantization in the Euclidean dS space [21] and the Gupta-Bleuler quantization [22] was studied and also the invariant observables under this symmetry rather than field operators were considered: difference of fields in different spacetime points [20]. In the approach of the present paper, the construction for the massless discrete series leads to quite different results from that of complementary series. The field operator so obtained turns out to be unique up to an overall complex constant and more importantly, it does not satisfy the canonical commutation relations. This QFT gives physically unacceptable effects: its coupling to an Unruh detector [23] leads to an infinite temperature. We then explore the limit towards the massless field starting from the canonical massive fields of the complementary series. The limit gives rise a Fock space which is larger than that obtained from the discrete UIR, the additional part being due to the zero modes. The representation is also deformed by the zero modes in a way which we explicitly determine. The zero modes however render the vacuum not normalizable. We then cure this problem by compactifying the scalar field on a circle $\Phi = \Phi + 2\pi L$. In order to get observables invariant under this rotation, we consider a vertex operator which is a dS invariant regularization of $\exp(i\Phi/L)$. We compute its two-point function and show that it presents a universal logarithmic infrared divergence for all dimensions. We thus recover in dS invariant way the results of [24] where these divergences were shown to lead to the restoration of broken symmetries.

The plan of the paper is as follows. From section 2 to section 5, we study the two dimensional case. In section 6, we generalize the results to arbitrary dimensions. In section 2, we describe the representations and the associated scalar product. We also examine the massless limit from the complementary series. In section 3, we construct the scalar field of the complementary series and determine the Bunch-Davies vacuum [25] and our generalization of the *in* and *out* vacua. Section 4 is devoted to the massless case and section 5 to the massless limit of the scalar field of the complementary series. Several appendices contain technical details used in the text.

2. The $\mathrm{SO}_0(1,2)$ group

We first concentrate on the two-dimensional de Sitter space dS_2 and the starting point of

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our approach is the UIR of the $SO_0(1,2)$ group, the group of linear transformations with determinant 1 which leaves $-(X^0)^2+(X^1)^2+(X^3)^2$ invariant and which is connected to the identity. This is the isometry group of dS_2. The arbitrary dimensional case, $SO_0(1,n)$ will be treated at the end of the paper. Let $\mathcal{J}$ be the generator of the rotation subgroup and $\mathcal{K}_1$ and $\mathcal{K}_2$ the two boosts. They verify the commutation relations:

$$[\,\mathcal{J},\,\mathcal{K}_1\,] = i\mathcal{K}_2\,, \qquad [\,\mathcal{J},\,\mathcal{K}_2\,] = -i\mathcal{K}_1\,, \qquad [\,\mathcal{K}_1,\,\mathcal{K}_2\,] = -i\mathcal{J}\,. \tag{2.1}$$

The quadratic Casimir operator

$$\mathcal{C} = \mathcal{J}^2 - \mathcal{K}_1^2 - \mathcal{K}_2^2\,, \tag{2.2}$$

commutes with all the generators and is constant on irreducible representation. Bargmann [5] classified the UIR according to the value of $\mathcal{C}$ and the eigenvalues m of $\mathcal{J}$:

(i) the principal series with $\mathcal{C} \le -\frac{1}{4}$, $\quad m = 0, \pm 1, \ldots$ or $m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$;

(ii) the complementary series with $-\frac{1}{4} < \mathcal{C} < 0$ and $m = 0, \pm 1, \ldots$;

(iii) the two discrete series $D_k^{\pm}$, with $\mathcal{C} = k(k+1)$ where k is a non-negative integer or half integer, and with $\pm m = k+1, k+2, \ldots$ where the $+$ $(-)$ sign characterizing D_k^+ (D_k^-)

In all these cases, writing $\mathcal{C} = s^2 - 1/4$, the three generators can be represented in the basis of the eigenstates of $\mathcal{J}$ as

$$\mathcal{J}\,|m) = m\,|m)\,, \qquad \mathcal{K}_\pm\,|m) = \pm i\left\{\, m \pm \left(s + \frac{1}{2}\right)\right\}|m \pm 1)\,. \tag{2.3}$$

where the raising and lowering operators are defined as $\mathcal{K}_\pm = \mathcal{K}_1 \pm i\mathcal{K}_2$. We have adopted these expressions because the action of $\mathcal{K}_\pm$ is linear in m. Therefore, they give rise to first order differential operators in the position representation, that is when acting on functions on the circle. Indeed, defining

$$|\phi) = \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} e^{-im\phi}\,|m)\,, \tag{2.4}$$

the action of the generators on an arbitrary ket $|\Psi)$ is given by

$$\begin{aligned}
(\phi|\,\mathcal{J}\,\Psi) &= -i\frac{d}{d\phi}(\phi|\Psi)\,, \qquad (2.5)\\
(\phi|\,\mathcal{K}_1\,\Psi) &= i\left\{\sin\phi\frac{d}{d\phi} + \left(s+\frac{1}{2}\right)\cos\phi\right\}(\phi|\Psi) = i\left[\frac{1}{2}\left\{\sin\phi, \frac{d}{d\phi}\right\} - s\cos\phi\right](\phi|\Psi)\,,\\
(\phi|\,\mathcal{K}_2\,\Psi) &= i\left\{-\cos\phi\frac{d}{d\phi} + \left(s+\frac{1}{2}\right)\sin\phi\right\}(\phi|\Psi) = -i\left[\frac{1}{2}\left\{\cos\phi, \frac{d}{d\phi}\right\} + s\sin\phi\right](\phi|\Psi)\,.
\end{aligned}$$

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In the sequel, it will be also useful to have the action of finite transformations. They are given by

$$
\begin{aligned}
(\phi|\, e^{i\theta\mathcal{J}}\, \Psi) &= (\phi+\theta|\Psi)\,, \\
(\phi|\, e^{i\rho\mathcal{K}_1}\, \Psi) &= (\cosh\rho + \sinh\rho\cos\phi)^{-s-1/2}\,(\phi_1|\Psi)\,, \\
(\phi|\, e^{i\lambda\mathcal{K}_2}\, \Psi) &= (\cosh\lambda + \sinh\lambda\sin\phi)^{-s-1/2}\,(\phi_2|\Psi)\,.
\end{aligned} \tag{2.6}
$$

where

$$
\begin{aligned}
\cos\phi_1 &= \frac{\cos\phi\cosh\rho + \sinh\rho}{\cosh\rho + \sinh\rho\cos\phi}\,, \qquad & \sin\phi_1 &= \frac{\sin\phi}{\cosh\rho + \sinh\rho\ \cos\phi}\,, \\
\cos\phi_2 &= \frac{\cos\phi}{\cosh\lambda + \sinh\lambda\ \sin\phi}\,, \qquad & \sin\phi_2 &= \frac{\sin\phi\cosh\lambda + \sinh\lambda}{\cosh\lambda + \sinh\lambda\ \sin\phi}\,.
\end{aligned} \tag{2.7}
$$

The choice adopted in eq. (2.3) yields the above simple expressions for the finite transformations and will be easily generalized to higher dimensions. However, it should be noticed that for the discrete and the complementary representations, the generators are not Hermitian with respect to the standard $\mathcal{L}^2$ scalar product, we denote $(\cdot|\cdot)$, for the discrete and the complementary representations. In fact, denoting $\mathcal{U}^{(s)}$ the representation (2.3) for some s of three generators of $\mathrm{SO}_0(1,2)$, we have

$$
(\Psi|\ \mathcal{U}^{(s)}\ \Psi') = (\mathcal{U}^{(-s^*)}\ \Psi|\Psi')\,, \tag{2.8}
$$

where $\mathcal{U}^{(s)}$ designates any of three generators of $\mathrm{SO}_0(1,2)$ in the representation (2.3). Hence the $\mathcal{U}^{(s)}$ are Hermitian with respect to the $\mathcal{L}^2$ scalar product only if s is purely imaginary, e.g. for the principal series. For the other two series we should define a new scalar product with respect to which the generators are Hermitian.

2.1 The complementary series

The complementary series is obtained by taking s real and belonging to the interval $-1/2 < s < 1/2$. It turns out that s and $-s$ are equivalent representations because there exists an intertwiner Q such that

$$
Q\,\mathcal{U}^{(s)} = \mathcal{U}^{(-s)}\,Q\,. \tag{2.9}
$$

This intertwiner defines a new scalar product, denoted $\langle\cdot|\cdot\rangle$, by

$$
\langle\Psi\,|\,\Psi'\rangle \equiv (\Psi\,|\,Q\,\Psi')\,, \tag{2.10}
$$

and with respect to which, the generators are Hermitian:

$$
\begin{aligned}
\langle\Psi\,|\,\mathcal{U}^{(s)}\,\Psi'\rangle &= (\Psi\,|\,Q\,\mathcal{U}^{(s)}\,\Psi') = (\Psi\,|\,\mathcal{U}^{(-s)}\,Q\,\Psi') \\
&= (\mathcal{U}^{(s)}\,\Psi\,|\,Q\,\Psi') = \langle\mathcal{U}^{(s)}\,\Psi\,|\,\Psi'\rangle\,.
\end{aligned} \tag{2.11}
$$

From the the action of the generators and the definition of Q we deduce the scalar product of states $|m)$ and $|n)$, i.e. the matrix element of Q, as

$$
\langle m|n\rangle = (m|Q\,n) = \langle m_0|m_0\rangle\,\frac{\Gamma\left(m_0+s+\frac{1}{2}\right)}{\Gamma\left(m_0-s+\frac{1}{2}\right)}\,\frac{\Gamma\left(m-s+\frac{1}{2}\right)}{\Gamma\left(m+s+\frac{1}{2}\right)}\,\delta_{m,n} \equiv Q_m\delta_{m,n}\,. \tag{2.12}
$$

For $-1/2 < s < 1/2$, this scalar product is positive and regular for all states $|m)$. For $0 < s < 1/2$ the scalar product of two position eigenvectors can be obtained by Fourier transformation. One finds [5]

$$
\begin{aligned}
Q(\phi-\phi') \equiv (\phi|Q\,\phi') &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\,Q_m \\
&= \frac{\langle m_0|m_0\rangle}{\sqrt{4\pi}}\,\frac{\Gamma\left(m_0+s+\frac{1}{2}\right)}{\Gamma\left(m_0-s+\frac{1}{2}\right)}\,\frac{\Gamma\left(-s+\frac{1}{2}\right)}{\Gamma(s)}\left(\sin^2\frac{(\phi-\phi')}{2}\right)^{s-\frac{1}{2}}.
\end{aligned}
\tag{2.13}
$$

This can be also obtained from the following differential equation:

$$
(\phi|\left(\mathcal{K}_+^{(-s)}\,Q - Q\,\mathcal{K}_+^{(s)}\right)\phi') = \left\{\sin\frac{\varphi}{2}\,\frac{d}{d\varphi} - \left(s-\frac{1}{2}\right)\cos\frac{\varphi}{2}\right\}Q(\varphi) = 0\,, \qquad \varphi = \phi-\phi'\,.
\tag{2.14}
$$

In the region $-1/2 < s < 0$, the above expression is singular as can be seen from the behavior of the last factor in the coincident point limit. To our knowledge this case has not been studied in the literature. To obtain an expression valid for all values of s in the complementary series we should consider the above kernel $Q(\phi-\phi')$ as a distribution. This distribution is given by

$$
\begin{aligned}
Q(\phi-\phi') &= \langle m_0|m_0\rangle\,\frac{\Gamma\left(m_0+s+\frac{1}{2}\right)}{\Gamma\left(m_0-s+\frac{1}{2}\right)}\,\frac{\Gamma\left(-s+\frac{1}{2}\right)}{(4\pi)^{\frac{1}{2}}\,\Gamma(s)}\times \\
&\times\lim_{\epsilon\to 0^+}\left[\left(\sin^2\frac{(\phi-\phi')}{2}+\frac{\epsilon}{2s}\right)\left(\sin^2\frac{(\phi-\phi')}{2}+\epsilon\right)^{s-\frac{3}{2}}\right].
\end{aligned}
\tag{2.15}
$$

This expression is now valid for $-1/2 < s < 1/2$ and reduces to eq. (2.13) for $0 < s < 1/2$. Furthermore, when $s \to 0$ it reduces, as it should, to

$$
Q(\phi-\phi') = \langle m_0|m_0\rangle\,\delta(\phi-\phi')\,,
\tag{2.16}
$$

which is the $\mathcal{L}^2$ kernel of the principal series. As we shall see below, this generalization of Q to negative values of s will be useful to study the massless case as a limiting procedure from the complementary series.

2.2 The massless case

The massless representations have a vanishing Casimir, i.e. $s^2 = 1/4$, and constitute the first discrete representations $D^{\pm}_{k=0}$ in the classification of UIR. The positive series D^+_0 has only states with positive m whereas the negative series D^-_0 with only negative m. Any state $|\Psi)$ in each series $D^{\pm}_0$ can thus be expanded as

$$
|\Psi) = \sum_{m=1}^{\infty} c_{\pm m}\,|\pm m)\,.
\tag{2.17}
$$

As can be seen from eq. (2.3) the expressions greatly simplify for $s = -1/2$:

$$
\mathcal{J}\,|m) = m\,|m)\,, \qquad \mathcal{K}_\pm\,|m) = \pm i\,m\,|m\pm 1)\,.
\tag{2.18}
$$

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They are Hermitian provided that the scalar product for $D_0^\pm$ is given by

$$\langle n|m\rangle = \frac{\langle m_\pm|m_\pm\rangle}{m_\pm}\, m\, \delta_{m,n}\,, \tag{2.19}$$

where m_+(resp. m_-) is an arbitrary positive(resp. negative) integer. It should be noticed that the zero mode $|m=0) \equiv i\,\mathcal{K}_\mp|\pm 1)$ has vanishing norm and so does not belong to the UIRs of $D_0^\pm$.

To be able to use the position representation of the generators acting on $D_0^\pm$, we consider the (larger) Hilbert space formed by the $\mathcal{L}^2$ functions on the circle. This space includes all modes m and allows the following decomposition of any element of $D_0^\pm$ as

$$\Psi(\phi) \equiv (\phi|\Psi) = \sum_{m=1}^{\infty} e^{\pm im\phi}\, c_{\pm m}\,. \tag{2.20}$$

In the position basis, the scalar product between two elements of $D_0^\pm$ is

$$\langle\Psi|\Psi'\rangle = -i\,\frac{\langle m_\pm|m_\pm\rangle}{m_\pm}\int_0^{2\pi} d\phi\,(\Psi|\phi)\,\frac{d}{d\phi}(\phi|\Psi')\,. \tag{2.21}$$

In this enlarged Hilbert space, the action of the three generators is given by

$$(\phi|\,\mathcal{J}\,\Psi) = -i\frac{d}{d\phi}(\phi|\Psi)\,, \quad (\phi|\,\mathcal{K}_1\,\Psi) = i\sin\phi\frac{d}{d\phi}(\phi|\Psi)\,, \quad (\phi|\,\mathcal{K}_2\,\Psi) = -i\cos\phi\frac{d}{d\phi}(\phi|\Psi)\,. \tag{2.22}$$

2.3 The massless limit

The limit $s \to -1/2$ of the complementary series leads at a first sight to three irreducible representations: D_0^+ with $m \geq 1$, D_0^- , with $m \leq -1$ and the trivial representation with $m = 0$. In fact, writing $s = -1/2 + \varepsilon$, the action of $\mathcal{K}_\mp$ on the normalized states with $m = \pm 1$ is

$$\mathcal{K}_\mp \frac{|\pm 1)}{\sqrt{\langle \pm 1|\pm 1\rangle}} = -i\sqrt{\varepsilon}\,\frac{|0)}{\sqrt{\langle 0|0\rangle}}\,, \qquad \mathcal{K}_\pm \frac{|0)}{\sqrt{\langle 0|0\rangle}} = \pm i\sqrt{\varepsilon}\,\frac{|\pm 1)}{\sqrt{\langle \pm 1|\pm 1\rangle}}\,. \tag{2.23}$$

These equations shows that the representation splits indeed into the three irreducible representations mentioned above. The scalar product however has a singular limit since eq. (2.12) gives

$$\langle 0|0\rangle = \frac{\langle m_0|m_0\rangle}{m_0}\,\varepsilon + \mathcal{O}(\varepsilon^2)\,. \tag{2.24}$$

The zero mode $|0)$ has thus a vanishing norm in the limit $s \to -1/2$, thereby recovering the results of the discrete series obtained in section 2.2.

3. Scalar field from complementary series

Let $\mathcal{H}_\Omega$ be the one-dimensional Hilbert space associated with the trivial representation $|\Omega)$ and $\mathcal{H}$ the Hilbert space carrying the UIR of the preceding section. The Fock space $\mathcal{F}$ is constructed from these two spaces in the following manner

$$\mathcal{F} = \mathcal{H}_\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes_{\rm sym} n}\,, \tag{3.1}$$

where the $n^{\rm th}$ term $\mathcal{H}^{\otimes_{\rm sym} n}$ is the representation obtained by taking the symmetrized tensor product of n copies of the UIR. Then, the creation and annihilation operators are defined as

$$
\begin{aligned}
a_m^\dagger|\Omega) = |m)\,, \qquad a_m|\Omega) = 0\,,\\
[\, a_m\,,\, a_n^\dagger\,] = Q_m\,\delta_{m,n}\,,
\end{aligned}
\tag{3.2}
$$

where Q_m is defined in eq. (2.12). The above relations can be used to define the creation and annihilation operators of the position basis:

$$
\begin{aligned}
a^\dagger(\phi)|\Omega) = |\phi)\,, \qquad a(\phi)|\Omega) = 0\,,\\
[\, a(\phi)\,,\, a^\dagger(\phi')\,] = Q(\phi-\phi')\,.
\end{aligned}
\tag{3.3}
$$

The (reducible) representation of the dS group on the Fock space is obtained from the irreducible representation by

$$
U = \sum_{m,n} \mathcal{U}_{mn}\, a_m^\dagger\, a_n\,, \tag{3.4}
$$

with $\mathcal{U}_{mn} = \langle m|\,\mathcal{U}\,|n\rangle/Q_m Q_n = (m|\,\mathcal{U}\,n)/Q_n$. Each n-particle sector of the Fock space is thus kept invariant under the action of the group transformations.

Using these operators, we shall now construct a local field, $\Phi(x)$ on dS space. We shall use the *global* coordinate system (t,θ), where t is the time coordinate and varies from $-\infty$ to $+\infty$ and θ is an angle coordinate. In this coordinate system, the metric is

$$
ds^2 = -dt^2 + \cosh^2 t\, d\theta^2. \tag{3.5}
$$

Later we shall also use the conformal time for which

$$
ds^2 = \sin^{-2}\!\eta\,(-d\eta^2 + d\theta^2)\,. \tag{3.6}
$$

The point with global coordinates $(0,0)$ is transported to (t,θ) by the following *ordered* sequence: first one acts with a boost generated by K_1 with parameter t to reach $(t,0)$, and then one reaches (t,θ) by a rotation with angle θ. Notice also that the point $(0,0)$ is left invariant by the boost generated by K_2.

Taking into account the non-commuting character of J and K_1 we have the central equation

$$
\Phi(t,\theta) = e^{-i\theta J} e^{itK_1}\,\Phi(0,0)\,e^{-itK_1} e^{i\theta J}\,. \tag{3.7}
$$

It determines the field operator $\Phi(t,\theta)$ from the operator $\Phi(0,0)$. The other important equation follows from the fact that $\Phi(0,0)$ must be invariant under transformations that leave the point $(0,0)$ invariant. In two dimensions only K_2 leaves $(0,0)$ invariant. Hence we impose

$$
[\, K_2,\, \Phi(0,0)\,] = 0\,. \tag{3.8}
$$

Eq. (3.7) and eq. (3.8) imply that $\Phi(t,\theta)$ obeys the Klein-Gordon equation with a mass squared term given by $M^2 = -s^2 + 1/4$, see I for the proof. Besides covariance, we also

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impose that our field is free of interactions. Hence we write it as a linear combination of creation and annihilation operators. In the Fourier basis we have

$$\Phi(0,0) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} c_m\, a_m^\dagger + c_m^*\, a_m\,. \tag{3.9}$$

The field operator is thus fully determined by the c-number constants c_m . The crucial point is that eq. (3.8) is not satisfied by arbitrary superpositions. Indeed imposing eq. (3.8) gives the following conditions:

$$\frac{c_{m+2}}{c_m} = -\frac{m+s+\frac{1}{2}}{m-s+\frac{3}{2}} = \frac{\gamma_{m+2}}{\gamma_m}\,, \tag{3.10}$$

with

$$\gamma_m = e^{im\frac{\pi}{2}} \frac{\Gamma\left(\frac{m}{2}+\frac{s}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{m}{2}-\frac{s}{2}+\frac{3}{4}\right)} \tag{3.11}$$

The solution here depends on *two* complex constants c_0 and c_1. These determine all other coefficients by

$$c_{2m} = c_0 \frac{\gamma_{2m}}{\gamma_0}\,, \qquad c_{2m+1} = c_1 \frac{\gamma_{2m+1}}{\gamma_1}\,. \tag{3.12}$$

In order to find how to interpret the set of covariant field operators characterized by c_0 and c_1, it is necessary to determine how these fields evolve in space-time.

3.1 Field operator in position basis

To determine the time evolution of the field, we work in the position basis

$$\Phi(0,0) = \int_0^{2\pi} d\phi\, \Psi_0(\phi)\, a^\dagger(\phi) + \Psi_0^*(\phi)\, a(\phi)\,. \tag{3.13}$$

with

$$\Psi_0(\phi) = (\phi|\, \Phi(0,0)\, \Omega)\,. \tag{3.14}$$

From eq. (3.8), we get the following differential equation:

$$(\phi|\, K_2\, \Phi(0,0)\, \Omega) \;=\; -i \left\{ \cos\phi \frac{d}{d\phi} - \left(s+\frac{1}{2}\right) \sin\phi \right\} \Psi_0(\phi) \;=\; 0\,. \tag{3.15}$$

The general solution is given by

$$\Psi_0(\phi) = A\, \Theta(\cos\phi)\, (\cos\phi)^{-s-\frac{1}{2}} + B\, \Theta(-\cos\phi)\, (-\cos\phi)^{-s-\frac{1}{2}}\,, \tag{3.16}$$

where Θ is the Heaviside step function and from the action of the dS transformation given in eq. (2.6) and the covariance of the field operator, we deduce the field operator at an arbitrary point in dS_2 as

$$\Phi(t,\theta) = \int_0^{2\pi} d\phi\, \Psi_{t,\theta}(\phi)\, a^\dagger(\phi) + \Psi_{t,\theta}^*(\phi)\, a(\phi)\,. \tag{3.17}$$

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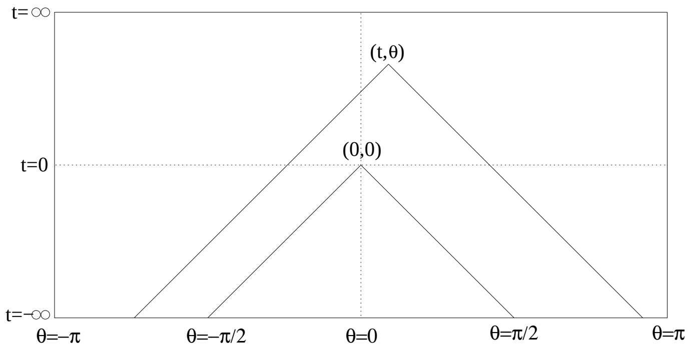

Figure 1: The Carter-Penrose diagram of the two dimensional dS space. We have represented the causal past of the point (t, θ).

with

$$\begin{aligned} \Psi_{t,\theta}(\phi) \,&=\, A\,\Theta(\cosh t\cos(\phi-\theta)+\sinh t)\,(\cosh t\cos(\phi-\theta)+\sinh t)^{-s-\frac{1}{2}} + \\ &\quad + B\,\Theta(-\cosh t\cos(\phi-\theta)-\sinh t)\,(-\cosh t\cos(\phi-\theta)-\sinh t)^{-s-\frac{1}{2}}\,. \end{aligned} \tag{3.18}$$

It is important to notice that although we got a first order differential equation, being singular at $\phi_\pm = \theta \pm \arccos(-\tanh t)$, its solution depends on two complex numbers (which can then be related to c_0 and c_1). Notice also that the interval $[\phi_-, \phi_+]$ corresponds to the region of space which is in the causal infinite past of (t, θ) (see, figure 1). From this we learn that the two-folded degeneracy of covariant field operators is deeply related to the causal structure of de Sitter space.

In the sequel it will be also useful to use the *regular* holomorphic and anti-holomorphic combinations:

$$\Psi_{t,\theta}(\phi) = C\,(\cosh t\cos(\phi-\theta)+\sinh t - i\epsilon)^{-s-\frac{1}{2}} +\; D\,(\cosh t\cos(\phi-\theta)+\sinh t + i\epsilon)^{-s-\frac{1}{2}}\,. \tag{3.19}$$

Writing, as usual, the field operator as

$$\Phi(t,\theta) = \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} c_m(t)\,e^{-im\theta}\,a^\dagger_m + c^*_m(t)\,e^{im\theta}\,a_m\,. \tag{3.20}$$

The coefficient $c_m(t)$ are thus given by

$$\begin{aligned} c_m(t) \,&=\, \sqrt{2\pi}\,(m|\,\Phi(t,0)\,\Omega) = \int_0^{2\pi} d\phi\, e^{-im\phi}\,\Psi_{t,0}(\phi) \\ &= \Gamma(-s+\tfrac{1}{2})\left(\frac{2\pi}{\cosh t}\right)^{\frac{1}{2}}\Big[(C+D)\,\mathrm{P}^s_{m-\frac{1}{2}}(-\tanh t) + \\ &\qquad + \left(C\,e^{-i\pi(s+\frac{1}{2})} + D\,e^{i\pi(s+\frac{1}{2})}\right)(-1)^m\,\mathrm{P}^s_{m-\frac{1}{2}}(\tanh t)\Big]\,, \end{aligned} \tag{3.21}$$

and from this, we get the relation between C, D and c_0, c_1 :

$$\begin{aligned} \frac{\Gamma\left(-\frac{s}{2}+\frac{3}{4}\right)}{\sqrt{\pi}\,\Gamma\left(-\frac{s}{2}+\frac{1}{4}\right)}\,c_0 \,&=\, \left(1+(e^{-i\pi})^{-s-\frac{1}{2}}\right)C + \left(1+(e^{i\pi})^{-s-\frac{1}{2}}\right)D\,, \\ \frac{\Gamma\left(-\frac{s}{2}+\frac{5}{4}\right)}{\sqrt{\pi}\,\Gamma\left(-\frac{s}{2}+\frac{3}{4}\right)}\,c_1 \,&=\, \left(1-(e^{-i\pi})^{-s-\frac{1}{2}}\right)C + \left(1-(e^{i\pi})^{-s-\frac{1}{2}}\right)D\,. \end{aligned} \tag{3.22}$$

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3.2 Commutation relations

In this subsection we show that the covariant fields, solutions of eq. (3.8), automatically obey canonical commutation relations up to an overall constant we shall determine.

Since $\gamma_m = \gamma_{-m}$, we have $c_m = c_{-m}$ for all m which implies that any covariant field is invariant under parity: $\theta \to -\theta$. In turn this implies the following commutation relation

$$[\,\Phi(0,\theta),\, \Phi(0,0)\,] = 0\,. \tag{3.23}$$

See I for the details of the proof. Next, we consider the commutator of the field with its conjugate momentum $\Pi = \partial_t \Phi$. Using

$$-i\,\partial_t \Phi(0,0) = [\,K_1,\, \Phi(0,0)\,] = \sum_{m=-\infty}^{\infty} \frac{i}{\sqrt{2\pi}} \left(m + s + \frac{1}{2}\right) c_m\, a^\dagger_{m+1} + \text{h.c.} \tag{3.24}$$

we obtain

$$h(\theta) = i\,[\,\Pi(0,0)\,, \Phi(0,\theta)\,] = \sum_{m=-\infty}^{\infty} \frac{i}{2\pi} \left(m + s - \frac{1}{2}\right) c^*_m c_{m-1}\, Q_m\, e^{im\theta} + \text{c.c.} \tag{3.25}$$

Now we use

$$\left(m + s + \frac{1}{2}\right) \gamma^*_m \gamma_{m-1}\, Q_m = -i\, 2^{-2s+1} \frac{\Gamma\left(m_0 + s + \frac{1}{2}\right)}{\Gamma\left(m_0 - s + \frac{1}{2}\right)} \langle m_0 | m_0 \rangle\,, \tag{3.26}$$

to get

$$h(\theta) = \mathrm{Re}\left(\frac{c_0\, c_1^*}{\gamma_0\, \gamma_1^*}\right) 2^{-2s+1} \frac{\Gamma\left(m_0 + s + \frac{1}{2}\right)}{\Gamma\left(m_0 - s + \frac{1}{2}\right)} \langle m_0 | m_0 \rangle\, \delta(\theta)\,. \tag{3.27}$$

The canonical commutation relations are thus satisfied if the coefficient of the Dirac function $\delta(\theta)$ is one. From this and the fact that $\gamma_0\gamma_1$ is purely imaginary, as can be seen from eq. (3.11), we get

$$c_1^* c_0 - c_1 c_0^* = i \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma\left(-s + \frac{1}{2}\right)} \frac{\Gamma\left(m_0 - s + \frac{1}{2}\right)}{\Gamma\left(m_0 + s + \frac{1}{2}\right)} \frac{1}{\langle m_0 | m_0 \rangle}\,. \tag{3.28}$$

In terms of C and D, this condition, using eq. (3.22), reads

$$|D|^2 - |C|^2 = \frac{\Gamma\left(s + \frac{1}{2}\right)}{\Gamma\left(-s + \frac{1}{2}\right)} \frac{\Gamma\left(m_0 - s + \frac{1}{2}\right)}{\Gamma\left(m_0 + s + \frac{1}{2}\right)} \frac{1}{\langle m_0 | m_0 \rangle} \frac{1}{8\pi\, \cos \pi s} \equiv N^2\,. \tag{3.29}$$

This quadratic form is invariant under SU$(1,1)$ transformations.

3.3 Hadamard requirement and Bunch-Davies vacuum

To understand the meaning of the set of covariant canonical fields, $\Phi_{c_0,c_1}(t,\theta)$, we consider the two-point function evaluated in the vacuum

$$G_{c_0,c_1}(t,\theta;t',\theta') \equiv (\Omega|\, \Phi_{c_0,c_1}(t,\theta)\, \Phi_{c_0,c_1}(t',\theta')\ \Omega)\,. \tag{3.30}$$

This observable, which parametrically depends on c_0 and c_1, can then be compared with the corresponding quantity obtained in the usual quantization of scalar fields on de Sitter space. This correspondence is most easily achieved when requiring that $G(0,\theta;0,0)$ possesses the Hadamard behavior in the coincidence point limit. This limit is governed by the high m behavior of the coefficients c_m. It coincides with that of flat space if

$$\frac{1}{2\pi}|c_m|^2\, Q_m \underset{|m|\to\infty}{\approx} \frac{1}{4\pi|m|}\,. \tag{3.31}$$

The asymptotic behaviors of $|\gamma_m|^2$ and Q_m are given by

$$|\gamma_m|^2 \underset{|m|\to\infty}{\approx} \left(\frac{m}{2}\right)^{2s-1}, \qquad Q_m \underset{|m|\to\infty}{\approx} m^{-2s}\,\frac{\Gamma\left(m_0+s+\frac{1}{2}\right)}{\Gamma\left(m_0-s+\frac{1}{2}\right)}\langle m_0|m_0\rangle\,. \tag{3.32}$$

The Hadamard condition thus gives

$$\frac{|c_0|^2}{|\gamma_0|^2} = \frac{|c_1|^2}{|\gamma_1|^2} = \frac{\Gamma\left(m_0-s+\frac{1}{2}\right)}{\Gamma\left(m_0+s+\frac{1}{2}\right)}\,\frac{2^{2(s-1)}}{\langle m_0|m_0\rangle}\,. \tag{3.33}$$

If we combine it with the canonical normalization, eq. (3.28), we get a unique solution we shall call Bunch-Davies (BD) [25]

$$\frac{c_0^{\rm BD}}{\gamma_0} = \frac{c_1^{\rm BD}}{\gamma_1} = \sqrt{\frac{\Gamma\left(m_0-s+\frac{1}{2}\right)}{\Gamma\left(m_0+s+\frac{1}{2}\right)}\,\frac{2^{2(s-1)}}{\langle m_0|m_0\rangle}}\,. \tag{3.34}$$

Using eq. (3.29), the BD condition simply reads

$$C^{\rm BD} = 0\,, \qquad D^{\rm BD} = N\,. \tag{3.35}$$

These equations show that the BD solution can be univoquely characterized by the canonical normalization and the anti-holomorphic behavior of the function Ψ_0 of eq. (3.19).

Using the above as the reference solution, the moduli space of canonical fields can be parametrized by

$$c_0 = c_0^{\rm BD}(\cosh\alpha + e^{i\beta}\sinh\alpha)\,, \qquad c_1 = c_1^{\rm BD}(\cosh\alpha - e^{i\beta}\sinh\alpha)\,. \tag{3.36}$$

The moduli space is $\mathrm{SU}(1,1)/\mathrm{U}(1)$. As explained for the principal series in I, it corresponds to the the space of Bogoliubov transformations which preserve de Sitter invariance and which relate two alpha-vacua [14]. Let us denote by $\Phi^+_{\rm BD}(x)$ the part of the BD field operator which contains only creation operators. Then, repeating the steps of section 5 of [4], the general field can be expressed as

$$\Phi_{\alpha,\beta}(x) = \cosh\alpha\,\Phi^+_{\rm BD}(x) + e^{i\beta}\sinh\alpha\,\Phi^+_{\rm BD}(\bar{x}) + \text{h.c.} \tag{3.37}$$

where $\bar{x} = (\pi-\eta, \theta+\pi)$ is the antipodal point to $x = (\eta,\theta)$.

We have reached the interpretation of our $\mathrm{SU}(1,1)/\mathrm{U}(1)$ set of covariant and canonical fields $\Phi_{\alpha,\beta}$. They provide an alternative description of the fact the canonical quantization of fields on de Sitter space supplemented by the requirement that the vacuum state be de

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Sitter invariant leads to a SU(1,1)/U(1) class of de Sitter invariant two-point functions. In the standard approach these two-point functions are viewed as expectation values built with *the* field operator in a class of states, the alpha-vacua. Here instead, the two-point functions are given by eq. (3.30), which is the expectation values of different field operators evaluated the unique vacuum state $|\Omega\rangle$ which carries the trivial representation.

3.4 *in* and *out* vacuum

One of the important physical difference between the fields in the principal and complementary series is their behavior at large t. In this limit the Klein-Gordon equation reads

$$(\partial_t^2 + \partial_t + M^2)\Phi = 0\,. \tag{3.38}$$

Its solutions are a combination of $e^{-t/2}\, e^{\pm i\mu t}$ with

$$\mu = \sqrt{M^2 - \frac{1}{4}}\,. \tag{3.39}$$

The principal series is characterized by $M^2 > \frac{1}{4}$ leading to solutions with oscillatory behavior and allowing the determination of *in* and *out* vacua as positive (proper) frequency modes in the remote past and future respectively [15]. For the complementary series we have instead $\mu = i\,s$ purely imaginary. Therefore the asymptotic solutions are exponentially decreasing in proper time, and the definition of *in* and *out* vacua no longer applies. The same problem arises for $M = 0$ where $\mu = -i/2$. In this case however, the conformal invariance of the Klein-Gordon equation leads to modes varying as $e^{\pm im\eta}$. The positive conformal frequency modes define the conformal vacuum which coincides with *in* and *out* asymptotic vacua since there is no frequency mixing. Here, we show that we can define *in* and *out* vacua for the complementary series as positive frequency solutions with respect to a time coordinate which interpolates between the proper time and the conformal time.

In the remote past, the coefficient $c_m(t)$ has the asymptotic behavior:

$$c_m(t) \underset{t\to-\infty}{\propto} e^{\left(s+\frac{1}{2}\right)t}\left\{1 + i\,\omega_m^{\text{in}}\,\left(2\,e^{-2st}\right)\right\}, \tag{3.40}$$

with

$$\omega_m^{\text{in}} = \frac{\Gamma(1+s)\,\Gamma(m-s+\frac{1}{2})}{2\,\Gamma(1-s)\,\Gamma(m+s+\frac{1}{2})}\,\frac{-\sin\pi s + i(\cos\pi s\cosh 2\alpha - \sin\beta\sinh 2\alpha)}{\cosh 2\alpha - \sin(\pi s+\beta)\,\sinh 2\alpha}. \tag{3.41}$$

Let us factorize the overall decreasing term(in $|t|$) and define a new time coordinate which in the infinite past is related to t as by

$$\eta_s = 2\,e^{-2st}. \tag{3.42}$$

It goes to zero in the remote past and reduces to the conformal time in this limit for $s = -1/2$. The last factor of $c_m(t)$ can be written as a plane wave to the first order in η_s:

$$e^{i\,\omega_m^{\text{in}}\,\eta_s} + \mathcal{O}(\eta_s{}^2)\,. \tag{3.43}$$

The *in* vacuum with respect to the η_s time coordinate is now defined by ω_m^{in} real and positive. The condition for ω_m^{in} to be real is

$$\tanh 2\alpha^{\text{in}}\, \sin\beta^{\text{in}} = \cos\pi s\,, \tag{3.44}$$

and the resulting value of ω_m^{in} is given by

$$\omega_m^{\text{in}} = \frac{\Gamma(1+s)\,\Gamma(m-s+\frac{1}{2})}{2\,\Gamma(1-s)\,\Gamma(m+s+\frac{1}{2})}\frac{\sqrt{\sin^2\beta-\cos^2\pi s}}{\cos(\pi s+\beta)}, \tag{3.45}$$

which is positive for β verifying eq. (3.44) and so $\pi(1/2+s) < \beta < \pi(1/2-s)$. Notice that we get a family of *in* vacua since β is an arbitrary angle in the interval $]\pi(1/2+s), \pi(1/2-s)[$. In the limit $s \to -1/2$, the frequency ω_m^{in} reduces as it should to $|m|$ which is the conformal frequency. The large m limit of ω_m^{in} varies as m^{-2s} which has the expected behavior when the complementary series joins the principal one that is for $s \to 0$.

The large future limit is given by

$$c_m(t) \underset{t\to\infty}{\propto} e^{-\left(s+\frac{1}{2}\right)t}\left\{1 + i\,\omega_m^{\text{out}}\,\left(-2\,e^{2st}\right)\right\}, \tag{3.46}$$

with

$$\omega_m^{\text{out}} = \frac{\Gamma(1+s)\,\Gamma(m-s+\frac{1}{2})}{2\,\Gamma(1-s)\,\Gamma(m+s+\frac{1}{2})}\,\frac{\sin\pi s\, + i(\cos\pi s\cosh 2\alpha + \sin\beta\sinh 2\alpha)}{\cosh 2\alpha + \sin(\pi s-\beta)\,\sinh 2\alpha}. \tag{3.47}$$

We demand that the time coordinate η_s is the remote future has an asymptotic expression in terms of t given by

$$\eta_s = \pi - 2e^{2st}, \tag{3.48}$$

so that $c_m(t)$ has an η_s dependence given by $e^{i\,\omega_m^{\text{out}}\,\eta_s}$. The *out* vacuum is such that ω_m^{out} is real and positive. The condition for ω_m^{out} to be real is

$$\tanh 2\alpha^{\text{out}}\, \sin\beta^{\text{out}} = -\cos\pi s\,. \tag{3.49}$$

and the resulting value of ω_m^{out} is given by

$$\omega_m^{\text{out}} = \frac{\Gamma(1+s)\,\Gamma(m-s+\frac{1}{2})}{2\,\Gamma(1-s)\,\Gamma(m+s+\frac{1}{2})}\frac{\sqrt{\sin^2\beta-\cos^2\pi s}}{\cos(\pi s-\beta)}, \tag{3.50}$$

which is positive for β verifying eq. (3.49) and so $-\pi(1/2-s) < \beta < -\pi(1/2+s)$. Notice that we get a family of *out* vacua since β is an arbitrary angle in the interval $]-\pi(1/2-s), -\pi(1/2+s)[$.

An explicit expression for the time coordinate η_s which gives the required asymptotic behavior for large and small t is

$$\tan\frac{\eta_s}{2} = e^{-2st}\,. \tag{3.51}$$

It reduces to the conformal time in the limit $s \to -1/2$.

The mean number of *out* quanta of momentum m in *in* vacuum is

$$\bar{n}_{\mathrm{out/in}} = \left| \cosh\alpha^{\mathrm{out}} \, \sinh\alpha^{\mathrm{in}} \, e^{i\beta^{\mathrm{in}}} - \cosh\alpha^{\mathrm{in}} \, \sinh\alpha^{\mathrm{out}} \, e^{i\beta^{\mathrm{out}}} \right|^2 \tag{3.52}$$

If we compute this between time-reversal *in* and *out* vacua with $\alpha^{\mathrm{out}} = \alpha^{\mathrm{in}}$, $\beta^{\mathrm{out}} = -\beta^{\mathrm{in}}$, then we get

$$\bar{n}_{\mathrm{out/in}} = \cosh^2(2\alpha^{\mathrm{in}}) \, \cos^2 \pi s = \frac{\sin^2\beta^{\mathrm{in}} \, \cos^2 \pi s}{\sin^2\beta^{\mathrm{in}} - \cos^2 \pi s} \tag{3.53}$$

Notice that when $\beta^{\mathrm{in}} = \pi/2$ the number of created quanta is minimal and is given by $\cot^2 \pi s$. It is easy to show that this is also the minimal number of created quanta when considering all *in* and *out* vacua.

In conclusion we have defined for the complementary series *in* and *out* vacua which belong to the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ moduli space of dS invariant vacua. Even though they are not unique for a given value of s they fulfill the criterion of being associated with positive frequency modes with respect to some asymptotic time parameter. The other unusual property is that this parameter now depends on s. These states could be relevant in certain inflationary models when considering nearly massless fluctuation modes with a value of s belonging to the complementary series but close to the discrete series which represents in four dimensions the massless minimally coupled field.

4. Massless scalar field from discrete series

Consider the UIR D_0^+ of section 2.2 and expand $\Phi_+(0,0)$ as

$$\Phi_+(0,0) = \frac{1}{\sqrt{2\pi}} \sum_{m=1}^{\infty} c_m \, a_m^\dagger + c_m^* \, a_m \,, \tag{4.1}$$

where

$$[\, a_m \,,\, a_n^\dagger \,] = \frac{\langle m_+ | m_+ \rangle}{m_+} \, m \, \delta_{m,n} \,, \qquad m \,, n \,, m_+ \in \mathbb{Z}_{>0} \,. \tag{4.2}$$

The locality condition $[\, K_2 \,,\, \Phi_+(0,0) \,] = 0$ gives

$$c_{2m} = 0 \,, \qquad (2m+1) c_{2m+1} = (-1)^m \, c_1 \,. \tag{4.3}$$

The commutator at equal time can now be readily calculated

$$\begin{aligned} [\, \Phi_+(0,\theta) \,,\, \Phi_+(0,0) \,] &= \frac{\langle m_+ | m_+ \rangle}{m_+} \frac{|c_1|^2}{2\pi} \sum_{m=0}^{\infty} \frac{e^{i(2m+1)\theta} - e^{-i(2m+1)\theta}}{2m+1} \\ &= i \, \frac{\langle m_+ | m_+ \rangle}{m_+} \, |c_1|^2 \, \mathrm{sgn}(\theta) \,. \end{aligned} \tag{4.4}$$

where $\mathrm{sgn}(\theta)$ is 2π-periodic, odd and equals $+1$ for $0 < \theta < \pi$ and -1 for $\pi < \theta < 2\pi$. The field is not a canonical one since the commutator at equal times does not vanish. The above commutation relation implies that

$$[\, \partial_\theta \Phi_+(0,\theta) \,,\, \Phi_+(0,0) \,] = 2i \, \frac{\langle m_+ | m_+ \rangle}{m_+} \, |c_1|^2 \, \delta(\theta) \,. \tag{4.5}$$

Using the fact that the time derivative at $t = 0$ is

$$\partial_t \Phi_+(0,0) = \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} (-1)^{m+1} \left(c_1\, a^\dagger_{2m+2} - c_1^*\, a_{2m+2} \right) . \tag{4.6}$$

we deduce that $\partial_t \Phi(0,0)$ and $\Phi(0,\theta)$ commute. Hence, Φ is not a canonical field.

In this we recover the fact that it is impossible to construct a canonical and covariant massless field on dS space. This is indeed in agreement with the result of Allen [14], who started with a massless canonical field and showed that it has no dS invariant two-point function. Here instead we have a well defined two-point function but we have lost the canonical commutation relations.

4.1 Non canonical massless field

In spite of the fact our field is non-canonical it is worth to further analyze its properties.

First, it obeys the conformally invariant equation $(\partial_\eta^2 - \partial_\theta^2)\Phi = 0$. This allows a simple determination of the field operator at an arbitrary point from the operator and its time derivative at $\eta = \pi/2$ and $\theta = 0$. A simple calculation yields

$$\Phi_+(\eta,\theta) = \frac{c_1}{\sqrt{2\pi}} \sum_{m=1}^{\infty} \frac{\sin(m\eta)}{m} \left(e^{-im\theta}\, a^\dagger_m + e^{im\theta}\, a_m \right) . \tag{4.7}$$

Second, its Wightman function $G(\eta,\theta\,;\,\eta',\theta') = (\Omega|\, \Phi_+(\eta,\theta)\, \Phi_+(\eta',\theta')\, \Omega)$ is well defined, thanks to the absence of the zero mode. Explicitly, one finds:

$$G(x;x') = \frac{\langle m_+|m_+\rangle}{m_+} \frac{|c_1|^2}{8\pi} \left[\ln\left| \frac{1+Z(x;x')}{1-Z(x;x')} \right| + i\pi\, \mathrm{sgn}(\theta-\theta')\, \Theta\left(1 - Z^2(x;x')\right) \right] . \tag{4.8}$$

where

$$Z(\eta,\theta\,;\,\eta',\theta') = \frac{\cos(\theta-\theta') - \cos\eta\, \cos\eta'}{\sin\eta\, \sin\eta'} = X^\mu X'_\mu\,. \tag{4.9}$$

This two-point function is dS invariant as can be easily seen from the embedding of dS_2 in a flat three dimensional space. The product $Z^{(1,2)}(x;x') = X^\mu X'_\mu$ is an invariant which is symmetric with respect to the exchange of the two spacetime points. It is related to the invariant distance $\sigma(x;x')$ between the two spacetime points by $Z(x;x') = \cos\{\sigma(x;x')\}$. When the vector $\epsilon_{\mu\nu\rho} X^\nu X'^\rho$ is timelike, an antisymmetric invariant is given by the sign of the time component. In this case we recover the function $\mathrm{sgn}(\theta-\theta')\,\Theta(1 - Z(x;x')^2)$ introduced above.[1]

[1]When $(X^\mu - X'^\mu)(X_\mu - X'_\mu)$ is negative, the antisymmetric invariants are given by the sign of the time component $X^0 - X'^0$ which is also the sign of the difference $t - t'$ or $\eta - \eta'$ when $Z(x;x') > 1$. Explicitly one has

$$s^{(1,2)}(x;x') = \mathrm{sgn}(\eta-\eta')\,\Theta(Z(x;x') - 1)\,. \tag{4.10}$$

When changing X'^μ to $-X'^\mu$, that is, when replacing x' by its antipodal point x'_A, gives the other invariant

$$s_A^{(1,2)}(x;x') = s^{(1,2)}(x;x'_A) = \mathrm{sgn}(\eta+\eta'-\pi)\,\Theta(-Z(x;x') - 1)\,. \tag{4.11}$$

Third, our field operator is closely related to the two-dimensional fields used in string theory [26]. Indeed, up to a normalization convention, our field coincides with that describing the coordinate of an open string with Dirichlet boundary conditions at both ends, and with the spatial and temporal coordinates interchanged: $\Phi_+(\eta,\theta) = X(\tau = -\theta, \sigma = \eta)$. At this point it is worth to notice that the interchange of the spatial and temporal coordinates gives a canonical massless field defined on AdS. In fact under this exchange, one recovers that the creation operators in eq. (4.7) have only positive frequencies. Moreover in that case, the canonical commutation relations fixes the normalization of c_1 in eq. (4.5).

Fourth, as in string theory, because of the conformal invariance, our field is covariant under a larger group of transformations. The generators of this group are

$$L_m = \sum_{n=1}^{\infty} a_n^\dagger \, a_{m+n} + \frac{1}{2}\sum_{n=1}^{m-1} a_n \, a_{m-n} \,, \qquad L_{-m} = L_m^\dagger \,, \qquad m \geq 0 \,, \tag{4.12}$$

and they act on the field as

$$\begin{aligned} [\, L_m \,,\, \Phi_+(\eta,\theta) \,] &= i \left(e^{-im(\theta+\eta)} \partial_{\theta+\eta} + e^{-im(\theta-\eta)} \partial_{\theta-\eta} \right) \Phi_+(\eta,\theta) \\ &= e^{-im\theta} \left(\sin m\eta \, \partial_\eta + i \cos m\eta \, \partial_\theta \right) \Phi_+(\eta,\theta) \,, \end{aligned} \tag{4.13}$$

and satisfy the Virasoro algebra with central charge $c = 1$:

$$[\, L_m \,,\, L_n \,] = (m-n)\, L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n} \,. \tag{4.14}$$

Notice that the generators of the dS group correspond to $L_{\pm 1} = \mp i \, K_\pm$ and $L_0 = J$. These three generators do not contain pair of creation or annihilation operators, as can be seen from eq. (4.12).

In spite of these well defined properties, the non-canonical character of our field shows up when considering the normal ordered operator of the energy-momentum tensor $T_{\mu\nu} = \, : \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial^\rho \Phi \partial_\rho \Phi :$ and the Hamiltonian operator:

$$H = \int_0^{2\pi} d\theta \; T_{\eta\eta} = \int_0^{2\pi} d\theta \; \left(T_{++} + T_{--} \right) \,, \tag{4.15}$$

where the subscripts $\pm$ are with respect to the light cone variables $x^\pm = \eta \pm \theta$. T_{++} and T_{--} give also the generators of Virasoro algebra, L_m, as

$$\frac{c^2}{2} L_m = \int_0^{2\pi} dx^+ \, e^{-imx^+} T_{++}(x^+) = \int_0^{2\pi} dx^- \, e^{imx^-} T_{--}(x^-) \,, \tag{4.16}$$

so one finds immediately $H = c^2 J$ and the Hamiltonian H generate rotations rather than time translations. In fact calculating the θ-translation generator in the Hamiltonian formalism as

$$P_\theta = \int_0^{2\pi} d\theta \; T_{\eta\theta} = \int_0^{2\pi} d\theta \; \left(T_{++} - T_{--} \right) \,, \tag{4.17}$$

we get identically $P_\theta = 0 \neq J$. This is due to the interchange of the role of time and space in the canonical commutation relation. We consider also the transition amplitude of a detector [23] with resonance frequency E, coupled linearly to Φ, and located at constant θ. This amplitude is proportional to

$$A(E) = \int_{-\infty}^{\infty} dt\, e^{-iEt}\, G(t,\theta;0,\theta) = \frac{|c|^2}{4E}\, \tanh\left(\frac{\pi}{2}E\right). \tag{4.18}$$

It is even in E so the detector has the same probability of loosing energy or gaining the same amount of energy. It cannot reach thermal equilibrium since the final steady state will be characterized by equally populated states (infinite temperature).

4.2 Parity invariant massless field

It should be noticed that our field is not invariant under parity. In what follows we shall construct a parity invariant field and show that the above sicknesses are not cured by this new field.

Consider first the scalar field constructed from the other discrete series D_0^-. Its expansion reads

$$\Phi_-(\eta,\theta) = \frac{c_{-1}}{\sqrt{2\pi}} \sum_{m=-\infty}^{-1} \frac{\sin(m\eta)}{m} \left(e^{-im\theta}\, a_m^\dagger + e^{im\theta}\, a_m\right). \tag{4.19}$$

where

$$[\, a_m\,,\, a_n^\dagger\,] = \frac{\langle m_-|m_-\rangle}{m_-}\, m\, \delta_{m,n}\,, \qquad m,\, n,\, m_- \in \mathbb{Z}_{<0}\,. \tag{4.20}$$

The generators of a new Virasoro algebra $\tilde{L}_m$'s are given by

$$\tilde{L}_m = \sum_{n=1}^{\infty} a_{-n}^\dagger\, a_{-m-n} + \frac{1}{2}\sum_{n=1}^{m-1} a_{-n}\, a_{-m+n}\,, \qquad \tilde{L}_{-m} = \tilde{L}_m^\dagger\,, \qquad m \geq 0\,. \tag{4.21}$$

The generators of the original dS group are $\tilde{L}_{\pm1} = \mp i\, K_\mp$ and $\tilde{L}_0 = -J$. The Virasoro algebra generated by $\tilde{L}_m$ have also a central charge $c = 1$, and act on the field as

$$\begin{aligned}[\, \tilde{L}_m\,,\, \Phi_-(\eta,\theta)\,] &= -i\left(e^{im(\theta+\eta)}\,\partial_{\theta+\eta} + e^{im(\theta-\eta)}\,\partial_{\theta-\eta}\right)\Phi_-(\eta,\theta)\\ &= e^{im\theta}\,(\sin m\eta\,\partial_\eta - i\cos m\eta\,\partial_\theta)\,\Phi_-(\eta,\theta)\,.\end{aligned} \tag{4.22}$$

Its two-point function is given by the complex conjugated of that of positive series (4.8), up to a normalization constant.

We can now construct a parity invariant field by taking the sum of the two previous fields with $c_1 = c_{-1} = c$, and with $\langle m_+|m_+\rangle/m_+ = -\langle m_-|m_-\rangle/m_-$ which we put to one for $m_\pm = \pm1$. Explicitly, one has

$$\begin{aligned}\Phi(\eta,\theta) &= \Phi_+(\eta,\theta) + \Phi_-(\eta,\theta)\\ &= \frac{c}{\sqrt{2\pi}} \sum_{m=1}^{\infty} \frac{\sin(m\eta)}{m} \left\{e^{-im\theta}\left(a_m^\dagger + a_{-m}\right) + e^{im\theta}\left(a_{-m}^\dagger + a_m\right)\right\}.\end{aligned} \tag{4.23}$$

The resulting field has a vanishing commutator for all spacetime points

$$[\, \Phi(\eta,\theta)\,,\, \Phi(\eta',\theta')\,] = 0\,. \tag{4.24}$$

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It therefore behaves as a classical field.

In addition, this is reflected in the Virasoro algebra generated by L_m induced from the L_m of Φ_+ and $\tilde{L}_m$ of Φ_-:

$$\mathsf{L}_m = L_m - \tilde{L}_{-m} \,. \tag{4.25}$$

which has a vanishing central charge. Even though the field has both positive and negative modes which correspond to left-moving and right-moving string oscillation modes, it has only one set of Virasoro algebra L_m. The other set of Virasoro algebra generated by $\tilde{\mathsf{L}}_m = L_m + \tilde{L}_{-m}$ does not leave the field covariant. Notice finally that the Wightman function is twice the symmetrical part of that of eq. (4.8). It is therefor real.

5. Massless limit of massive field

Given that we obtained a set of canonical fields when dealing with the complementary series and no canonical field for the discrete series, it is interesting to consider the massless limit from the complementary series: $\varepsilon = s + 1/2 \to 0$.

From eq. (3.21) and eq. (3.34), the BD coefficients are given by

$$c_m^{\mathrm{BD}}(\eta) = \frac{1}{\sqrt{2}} \begin{cases} \varepsilon^{-1} + \ln(2\sin\eta) - \frac{1}{2} + i\left(\eta - \frac{\pi}{2}\right) + \mathcal{O}(\varepsilon) & ;\ m = 0 \\ |m|^{-1}\, e^{i|m|\eta} + \mathcal{O}(\varepsilon) & ;\ m \neq 0 \end{cases}, \tag{5.1}$$

where we chose the normalization $m_0 = 1 = \langle 1|1\rangle$. The commutation relations and the scalar product behave as

$$[\, a_m \,,\, a_m^\dagger \,] = \langle m|m\rangle = \begin{cases} \varepsilon(1-\varepsilon)^{-1} & ;\ m = 0 \\ |m| + \mathcal{O}(\varepsilon) & ;\ m \neq 0 \end{cases}. \tag{5.2}$$

From eq. (5.1), we deduce the behavior of the BD field

$$\begin{aligned} \Phi_{\mathrm{BD}}(\eta,\theta) \,=\, & \frac{1}{2\sqrt{\pi}} \left[\left(\frac{1}{\varepsilon} + \ln(2\sin\eta) - \frac{1}{2} \right) \left(a_0^\dagger + a_0 \right) + i\left(\eta - \frac{\pi}{2} \right) \left(a_0^\dagger - a_0 \right) \right] + \\ & + \frac{1}{2\sqrt{\pi}} \sum_{m\neq 0} \frac{1}{|m|} \left(e^{i|m|\eta - im\theta}\, a_m^\dagger + e^{-i|m|\eta + im\theta}\, a_m \right) \,+\, \mathcal{O}(\varepsilon) \,. \end{aligned} \tag{5.3}$$

In the limit $\varepsilon \to 0$, the zero mode contains a ε^{-1} divergence whereas the commutation relation of a_0 and $a_0^\dagger$ vanishes as ε. The two combinations which appear in the zero mode are

$$q = \frac{1}{2\sqrt{\pi}\,\varepsilon} \left(a_0^\dagger + a_0 \right) \,, \qquad p = i\sqrt{\pi}(1-\varepsilon) \left(a_0^\dagger - a_0 \right) \,. \tag{5.4}$$

They obey a canonical commutation relation $[\, q \,,\, p \,] = i$. The zero mode part of the BD field is thus expressed as

$$\int_0^{2\pi} \frac{d\theta}{2\pi}\, \Phi_{\mathrm{BD}}(\eta,\theta) = q \,+\, p\, \frac{\eta - \frac{\pi}{2}}{2\pi} \,+\, \mathcal{O}(\varepsilon) \,, \tag{5.5}$$

which has a finite limit.

For finite ε the vacuum $|\Omega)$ is annihilated by $a_0 = \sqrt{\pi}\varepsilon q + i\{2\sqrt{\pi}(1-\varepsilon)\}^{-1} p$. When ε goes to zero with finite q and p, this condition reduces to $p\,|\Omega) = 0$ [18]. Such a state cannot be normalized as is the case in the quantum mechanics for a one-dimensional harmonic oscillator in the limit of vanishing frequency. The resulting Fock space is therefore the tensor product of the Fock space we obtained in section 4.2 and the Hilbert space carrying a representation of the commutator $[\,q\,,\,p\,] = i$.

In the $\varepsilon \to 0$ limit the zero mode operator p appears also in the generators $K_\pm$ of the dS group

$$K_+ \;=\; \frac{p}{2\sqrt{\pi}}\left(a_1^\dagger + a_{-1}\right) + i \sum_{m\neq 0,-1} a_{m+1}^\dagger\, a_m\,. \tag{5.6}$$

The condition that the vacuum be dS invariant leads again to $p\,|\Omega) = 0$. It is important to notice that we no longer are in the general framework we described in section 3 where the Fock space and the generators of the dS group are constructed only from the irreducible representations. The massless limit leads indeed to a larger Fock space and modified generators: In the subspace where $p = 0$, we recover the generators constructed from the UIR of the discrete series. When $p \neq 0$, the number of particles is no longer dS invariant, i.e. the subspace with N particles is no longer invariant under the dS transformations.

In the previous section when considering the massless field from discrete series, we found a unique field operator once we require its covariance. That expression differs from the above $\Phi_{\rm BD}$. This follows from the presence of the zero mode operators in the generators of the dS group. It is therefore of interest to study how the zero mode operators transform under the SU(1,1) group. Using eq. (3.37), the massless limit of the general field operator is

$$\begin{aligned}\Phi_{\alpha,\beta}(\eta,\theta) \;=\; & q_{\alpha,\beta} \;+\; p_{\alpha,\beta}\,\frac{\eta-\frac{\pi}{2}}{2\pi} \;+ \\ & + \frac{1}{2\sqrt{\pi}}\sum_{m\neq 0}\frac{1}{|m|}\left(\cosh\alpha\, e^{i|m|\eta} + \sinh\alpha\, e^{i\beta} e^{-i|m|\eta}\right) e^{-im\theta} a_m^\dagger \;+\; \text{h.c.}\end{aligned} \tag{5.7}$$

where the zero mode operators $q_{\alpha,\beta}$ and $p_{\alpha,\beta}$ are given by

$$\begin{aligned} q_{\alpha,\beta} \;&=\; \frac{1}{2\sqrt{\pi}\,\varepsilon}\left\{\left(\cosh\alpha + e^{i\beta}\sinh\alpha\right) a_0^\dagger + \text{h.c}\right\}, \\ p_{\alpha,\beta} \;&=\; i\sqrt{\pi}(1-\varepsilon)\left\{\left(\cosh\alpha - e^{i\beta}\sinh\alpha\right) a_0^\dagger - \text{h.c}\right\}, \end{aligned} \tag{5.8}$$

The dS generators now read

$$K_+ \;=\; \frac{p_{\alpha,\beta}}{2\sqrt{\pi}}\,\frac{a_1^\dagger + a_{-1}}{\cosh\alpha + \cos\beta\sinh\alpha} \;+\; i \sum_{m\neq 0,-1} a_{m+1}^\dagger\, a_m\,. \tag{5.9}$$

Notice that the generators depend explicitly on α and β. Notice also that the dS invariant vacuum must satisfy $p_{\alpha,\beta}\,|\Omega) = 0$ which is the same condition as $p\,|\Omega) = 0$. Notice finally that unless $\beta = \pi$, in the limit $\alpha \to +\infty$ the zero modes cancel out from the generators and the field operator which for $\beta = 0$ reads

$$\Phi_{\alpha,0}(\eta,\theta) = q_{\alpha,0} + p_{\alpha,0}\,\frac{\eta - \frac{\pi}{2}}{2\pi} + \\ + \frac{1}{2\sqrt{\pi}} \sum_{m\neq 0} \frac{1}{|m|} \left(i\, e^{\alpha} \sin|m|\eta + e^{-\alpha}\cos m\eta\right) e^{-im\theta} a_m^\dagger + \text{h.c.} \tag{5.10}$$

tends to the non canonical commuting field of eq. (4.23).

We finally notice that the energy momentum tensor given in the light cone coordinates by $T_{++} = \,:\partial_+\Phi\,\partial_+\Phi:$ and $T_{--} = \,:\partial_-\Phi\,\partial_-\Phi:$ now generates two copies of Virasoro algebra:

$$L_m = \int_0^{2\pi} dx^+\; e^{imx^+}\, T_{++}(x^+)\,, \qquad \tilde{L}_m = \int_0^{2\pi} dx^-\; e^{imx^-}\, T_{--}(x^-)\,. \tag{5.11}$$

The zero mode generators L_0 and $\tilde{L}_0$ define the rotation generator:

$$J = \tilde{L}_0 - L_0 = \sum_{m\geq 1} a_m^\dagger\, a_m - a_{-m}^\dagger\, a_{-m}\,, \tag{5.12}$$

and a Hamiltonian:

$$H = L_0 + \tilde{L}_0 = \frac{p^2}{4\pi} + \sum_{m=1}^{\infty} \Big[\cosh 2\alpha \left(a_m^\dagger\, a_m + a_{-m}^\dagger\, a_{-m}\right) + \\ + \sinh 2\alpha \left(e^{i\beta}\, a_m^\dagger\, a_{-m}^\dagger + e^{-i\beta}\, a_m\, a_{-m}\right) \Big]\,. \tag{5.13}$$

which reproduces the correct time evolution, in agreement with the fact that the field is canonical.

5.1 Vertex operator and massless two-point function

As we saw in the previous section, the dS invariant vacuum is not normalizable since it is the solution to $p\,|\Omega) = 0$. An easy way to regularize this infinity is to compactify the scalar field on a circle, that is to identify Φ and $\Phi + 2\pi L$ where L is the radius of the circle. This amounts to compactify the zero mode q and so p has discrete eigenvalues n/L with $n \in \mathbb{Z}$ and its eigenmodes are normalizable. Observables should be invariant under $\Phi \to \Phi + 2\pi L$. This excludes the scalar field Φ as an observable but includes derivatives of Φ or a covariant regularization of $\exp(i\Phi/L)$ which we will call the vertex operator V.

We define V at the origin of spacetime as the normal ordered exponential, that is

$$V(0) = \,: e^{\frac{i}{L}\Phi(0)} :\,, \tag{5.14}$$

where $:\cdot:$ is normal-ordering prescription and it does not affect the zero mode. It can be obtained from the massive fields as

$$V(0) = \lim_{\varepsilon\to 0} e^{\frac{i}{L}\Phi_\varepsilon^+(0)}\, e^{\frac{i}{L}\Phi_\varepsilon^-(0)}\, e^{\frac{1}{4\pi L^2}[c_0\, a_0^\dagger\,,\, c_0^*\, a_0]}\,. \tag{5.15}$$

Here, Φ_ε^+ (resp. Φ_ε^-) denotes the part of the massive field depending on creation (resp. annihilation) operators. This shows that $V(0,0)$ commutes with K_2 because $\Phi^\pm$ do and the last factor is a c-number. And we can translate with the dS transformations

$$V(t,\theta) = e^{-iJ\theta}\, e^{iK_1 t}\, V(0,0)\, e^{-iK_1 t}\, e^{iJ\theta} \;=\lim_{\varepsilon\to 0} e^{\frac{i}{L}\Phi_\varepsilon^+(t,\theta)}\, e^{\frac{i}{L}\Phi_\varepsilon^-(t,\theta)}\, e^{\frac{1}{4\pi L^2}[\,c_0\, a_0^\dagger\,,\,c_0^*\, a_0\,]}\,, \tag{5.16}$$

and this is an dS covariant definition of the vertex operator. If we decompose the massless BD field as $\Phi(\eta,\theta) = \phi^0(\eta,\theta) + \phi^+(\eta,\theta) + \phi^-(\eta,\theta)$ with

$$\phi^0(\eta,\theta) = q + p\frac{\eta - \frac{\pi}{2}}{2\pi}\,,\;\; \phi^+(\eta,\theta) = \frac{1}{2\sqrt{\pi}}\sum_{m\neq 0}\frac{1}{|m|}e^{i|m|\eta - im\theta}\, a_m^\dagger\,,\;\; \phi^-(\eta,\theta) = \left(\phi^+(\eta,\theta)\right)^\dagger\,, \tag{5.17}$$

then the vertex operators defined in eq. (5.16) reads

$$V(\eta,\theta) \;=\; e^{\frac{i}{L}\phi^+(\eta,\theta)}\, e^{\frac{i}{L}\phi^-(\eta,\theta)}\, e^{\frac{i}{L}\phi^0(\eta,\theta)}\, (\sin\eta)^{\frac{1}{4\pi L^2}} \tag{5.18}$$

It is important to notice that this definition of the vertex operator is not same as the usual one $:e^{\frac{i}{L}\Phi(x)}:$ which was considered in [27]. The last factor $(\sin\eta)^{1/4\pi L^2}$ in eq. (5.18) is necessary for the dS covariance.

The two-point function of vertex operators is readily calculated and is given by

$$(\Omega|\, V^\dagger(x)\, V(x')\;\Omega) = \exp\left[-\frac{1}{4\pi L^2}\,\log\left\{2\left(1 - \tilde{Z}(\bar{x};\bar{x}')\right)\right\}\right]\,, \tag{5.19}$$

where $\tilde{Z}(x;x') = Z(x;x') + i\,\mathrm{sgn}(t-t')\,\epsilon$. Since $\log z$ has a branch cut on the negative real axis, the term $i\,\epsilon\,\mathrm{sgn}(t-t')$ contributes only when $Z(x;x') > 1$ and thus is equivalent to a dS invariant quantity $i\,\epsilon\, s^{(1,2)}(x,x')$ in eq. (4.10). Furthermore it is related to the massive two-point function which has a divergence in $(4\pi\varepsilon)^{-1}$ by

$$\lim_{\varepsilon\to 0}\,\exp\left[\frac{1}{L^2}\left\{(\Omega|\,\Phi_\varepsilon(x)\,\Phi_\varepsilon(x')\,\Omega) - \frac{1}{4\pi}\left(\frac{1}{\varepsilon} + 2\ln 2\right)\right\}\right]. \tag{5.20}$$

Our regularization isolates the divergent piece of the two-point function and leaves a dS invariant result.

6. Arbitrary dimension

In this section we generalize to arbitrary dimensions the approach we have used in two dimensions. The scalar UIR will be easily generalized by their realizations with functions on the $(n-1)$-sphere, S^{n-1}.

The n-dimensional de Sitter space, dS_n is described by the hyperboloid in $(n+1)$-dimensional Minkowski space $\mathbb{R}^{1,n}$:

$$\eta_{AB}X^A X^B \;=\; 1\,, \tag{6.1}$$

where $\eta_{AB} = diag(-1,1,\ldots,1)$. In the following we use the index notation:

$$\begin{aligned} A,B,C,D &= 0,1,\ldots,n\,; & I,J &= 1,2,\ldots,n\,; \\ \mu,\nu &= 0,1,\ldots,n-1\,; & i,j,k &= 1,2,\ldots,n-1\,. \end{aligned} \tag{6.2}$$

6.1 The complementary series of $\mathbf{SO}_0(1,n)$

The isometry group of dS_n is $\mathrm{SO}_0(1,n)$. It is the group of special orthogonal transformations continuously connected to the identity which leaves eq. (6.1) invariant. The generators of $\mathrm{SO}_0(1,n)$ verify the following algebra:

$$[\,\mathcal{M}_{AB},\,\mathcal{M}_{CD}\,] \;=\; -i\;(\,\eta_{AC}\mathcal{M}_{BD}-\eta_{AD}\mathcal{M}_{BC}-\eta_{BC}\mathcal{M}_{AD}+\eta_{BD}\mathcal{M}_{AC}\,)\,. \tag{6.3}$$

The scalar representations of $\mathrm{SO}_0(1,n)$ can be realized on the space of functions on S^{n-1}, which is conveniently parametrized by a vector $\vec{\zeta}$ in $\mathbb{R}^n$ subject to $|\vec{\zeta}|=1$. The action of the generators of $\mathrm{SO}_0(1,n)$ in the representation labeled by s are given by

$$\begin{aligned}(\,\vec{\zeta}\,|\,\mathcal{M}_{IJ}\,\Psi) &= i\left(\zeta_I\frac{\partial}{\partial\zeta^J}-\zeta_J\frac{\partial}{\partial\zeta^I}\right)(\,\vec{\zeta}\,|\,\Psi)\,,\\ (\,\vec{\zeta}\,|\,\mathcal{M}_{I0}\,\Psi) &= \left\{\zeta^J\mathcal{M}_{IJ}+i\left(s+\frac{n-1}{2}\right)\zeta_I\right\}(\,\vec{\zeta}\,|\,\Psi)\,,\end{aligned} \tag{6.4}$$

and the quadratic Casimir is constant in this representation space and is given by

$$\mathcal{C}=\frac{1}{2}\sum_{A,B}\mathcal{M}_{AB}\,\mathcal{M}^{AB}=s^2-\frac{(n-1)^2}{4}\,, \tag{6.5}$$

with $-(n-1)/2<s<(n-1)/2$ for the complementary series. Physically, the value of the Casimir is the opposite of the mass squared of the particle $M^2=(n-1)^2/4-s^2$. It will also be useful to have the action under finite transformations, we have

$$\begin{aligned}(\,\vec{\zeta}\,|\,e^{i\theta\mathcal{M}^{IJ}}\,\Psi) &= (\,\vec{\zeta'}\,|\Psi)\,,\\ (\,\vec{\zeta}\,|\,e^{i\omega\mathcal{M}^{I0}}\,\Psi) &= (\cosh\omega+\zeta^I\sinh\omega)^{-s-\frac{n-1}{2}}(\,\vec{\zeta''}\,|\Psi)\,,\end{aligned} \tag{6.6}$$

where

$$\begin{aligned}\vec{\zeta'} &= e^{i\theta M^{IJ}}\vec{\zeta}\,,\\ \vec{\zeta''} &= \left(\frac{\zeta^1}{\cosh\omega+\zeta^I\sinh\omega},\,\dots\,,\frac{\sinh\omega+\zeta^I\cosh\omega}{\cosh\omega+\zeta^I\sinh\omega},\,\dots\,,\frac{\zeta^n}{\cosh\omega+\zeta^I\sinh\omega}\right),\end{aligned} \tag{6.7}$$

and where M^{IJ} is the representation of $\mathcal{M}^{IJ}$ on the vector space $\mathbb{R}^n$.

For purely imaginary s, this representation is unitary with respect to the $\mathcal{L}^2$ scalar product, $(\cdot|\cdot)$, and this is the principal series. For real s, the scalar product with respect to which this representation is unitary is, as in two dimensions, $\langle\cdot|\cdot\rangle\equiv(\cdot|Q\,\cdot)$ where Q is the intertwiner operator defined in eq. (2.9). From $(\,\vec{\zeta}\,|\,\mathcal{M}^{(-s)}_{AB}Q\,\vec{\zeta'}\,)=(\,\vec{\zeta'}\,|\,\mathcal{M}^{(-s)}_{AB}Q\,\vec{\zeta}\,)^*$, the intertwiner is determined up to a normalization:

$$Q(\vec{\zeta}\cdot\vec{\zeta'}\,)\equiv(\,\vec{\zeta}\,|\,Q\,\vec{\zeta'}\,)=Q_0\,\frac{\Gamma\big(s+\frac{n-1}{2}\big)}{(2\pi)^{\frac{n-1}{2}}\,2^s\,\Gamma(s)}\left(1-\vec{\zeta}\cdot\vec{\zeta'}\right)^{s-\frac{n-1}{2}}\,. \tag{6.8}$$

This scalar product is well defined if the value of s is restricted to $0<s<(n-1)/2$ and this is the complementary series.

The generalization of the Fourier basis valid in the two dimensional case is here provided by the spherical harmonics in n dimensions. These are conveniently defined from the set of homogeneous polynomials of degree L in n variables X_I which are harmonic. It can be shown that there are $N(n,L) = (2L+n-2)(L+n-3)!/L!(n-2)!$ independent harmonic and homogeneous polynomials. Their restriction on the unit sphere defines the spherical harmonics $S_{L,k}(\vec{\zeta}) = (\,\vec{\zeta}\,|L,k)$ with $L = 0,1,\ldots,$ and $k = 0,1,\ldots,N(n,L)$.

Notice that $\mathcal{M}_{IJ}\mathcal{M}^{IJ}$ is constant on the set of harmonic and homogeneous polynomials of degree L and is given by $L(L+n-2)$. It is easy to verify that the operators $\mathcal{M}_{I\pm}$ defined by

$$(\,\vec{\zeta}|\mathcal{M}_{I+}|L,k) = i\left(\zeta_I - \frac{1}{n+2L-2}\partial_I\right)(\,\vec{\zeta}\,|L,k)\,, \qquad (\,\vec{\zeta}|\mathcal{M}_{I-}|L,k) = i\,\partial_I(\,\vec{\zeta}\,|L,k)\,, \tag{6.9}$$

change the degree of L by ± 1. Then the boost can be deduced from the expression of the generators as

$$\mathcal{M}_{I0}\,|L,k) = \left\{\left(L+s+\frac{n-1}{2}\right)\mathcal{M}_{I+} + \frac{s-L-\frac{n-3}{2}}{L+n-1}\,\mathcal{M}_{I-}\right\}|L,k)\,, \tag{6.10}$$

thus the boost generators $\mathcal{M}_{I0}$ change the degree L of $|L,k)$ by ± 1 while the rotation generators $\mathcal{M}_{IJ}$ leave the degree L unchanged. Since the intertwiner Q commutes with the rotations, it is of the form $Q\,|L,k) = Q_L\,|L,k)$ where Q_L are c-numbers determined from the commutation with $\mathcal{M}_{I0}$. Using eq. (6.10), we get

$$Q_L = \langle L_0|L_0\rangle\,\frac{\Gamma(s+\frac{n-1}{2}+L_0)\,\Gamma(-s+\frac{n-1}{2}+L)}{\Gamma(-s+\frac{n-1}{2}+L_0)\,\Gamma(s+\frac{n-1}{2}+L)}\,. \tag{6.11}$$

Here we introduced a reference state L_0 with a norm $\langle L_0|L_0\rangle$. In the following we shall take $L_0 = 1$ and $\langle\,1\,|\,1\,\rangle = 1$, this also fixes the normalization constant Q_0 in eq. (6.8) by $Q_0 = (s+\frac{n-1}{2})/(-s+\frac{n-1}{2})$. Notice that the expression (6.11) is valid for s positive and negative in the interval $]-(n-1)/2,\,(n-1)/2[$ whereas the expression (6.8) applies only for positive s. The above expression of the intertwiner in spherical harmonic basis can be also obtained also from eq.(6.8) as is shown in appendix A.

6.2 The massless representation of $\mathbf{SO}_0(1,n)$

It is the first member of the discrete series with a vanishing quadratic Casimir operator. It can be realized on functions on the sphere S^{n-1} with vanishing zero modes. A basis is thus given by the spherical harmonics with $L \geq 1$. The action of the generators is given by eq. (6.4) with $s = (n-1)/2$. The rotations do not change the degree of homogeneity and the action of the boosts is given by

$$\mathcal{M}_{I0}\,|L,k) = \left\{(L+n-1)\,\mathcal{M}_{I+} + \frac{L-1}{L+n-1}\,\mathcal{M}_{I-}\right\}|L,k)\,, \tag{6.12}$$

where $\mathcal{M}_{I\pm}$ were defined in eq. (6.9). For $L = 1$, the right-hand side is of degree $L = 2$, the set of harmonics with strictly positive degree is thus invariant by the action of the generators. The generators will be Hermitian with respect to the scalar product $(\cdot|Q\,\cdot)$, with Q deduced as for the complementary series with $s = (n-1)/2$:

$$Q_L = \frac{(n-1)!(L-1)!}{(L+n-2)!}\,. \tag{6.13}$$

Notice that the limit $s \to (n-1)/2$ of the complementary series decomposes into an invariant state ($L = 0$) and the massless representation. The limit is singular because the norm of the zero mode varies as $\Gamma(s - \frac{n-1}{2})$ which becomes infinite.

As noticed before, an equivalent representation is provided by $s = -(n-1)/2$. The scalar product is now given by

$$Q_L = \frac{(L+n-2)!}{(n-1)!(L-1)!}\,. \tag{6.14}$$

The limit $s \to -(n-1)/2$ of the norm of the zero mode of the complementary series is now zero making this description of the massless representation more convenient. However in this case, the action of the boosts on the $L = 1$ state generates the zero norm $L = 0$ state:

$$\mathcal{M}_{I0}\,|1,k) = \left(\mathcal{M}_{I+} - \frac{n-1}{n}\,\mathcal{M}_{I-}\right)|1,k)\,. \tag{6.15}$$

6.3 Scalar field from the complementary series

We choose the origin in dS_n to have the embedding coordinates $X_o^A = (0, 0, \ldots, 0, 1)$. The field on any point of dS_n, of coordinates $X^A = \Lambda^A{}_B X_o^B$, can be deduced from the field at the origin $\Phi(X_o)$ by

$$\Phi(X) = U(\Lambda)\,\Phi(X_o)\,U(\Lambda)^{-1}\,, \tag{6.16}$$

where Λ is an element of $\mathrm{SO}_0(1,n)$.

In the global coordinate system:

$$\begin{aligned} X^0 &= \sinh t\,, \\ X^I &= \cosh t\,\xi^I\,, \qquad \vec{\xi} \in S^{n-1}\,, \\ \xi^I &= \mathrm{R}^I{}_J\,\xi_o^J\,, \qquad \vec{\xi}_o = (0,\ldots,0,1)\,, \end{aligned} \tag{6.17}$$

where R is a element of $\mathrm{SO}(n)$ subgroup. The metric reads

$$ds^2 = -dt^2 + \cosh^2 t\, d\Omega^2(\vec{\xi})\,. \tag{6.18}$$

Therefore the point X_o can be transported to any point X by a boost followed by a rotation. This implies that eq. (6.16) can be written as

$$\Phi(X) = U(\mathrm{R})\,e^{itM^{0n}}\,\Phi(X_o)\,e^{-itM^{0n}}\,U(\mathrm{R})^{-1}\,. \tag{6.19}$$

The origin $X_o = (0, \vec{\xi}_o)$ is invariant under the action of the subgroup $\mathrm{SO}_0(1, n-1)$ generated by $M_{\mu\nu}$. To construct a local field, we thus require

$$[\,M_{\mu\nu},\,\Phi(0,\vec{\xi}_o)\,] = 0\,. \tag{6.20}$$

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From the UIR of $\mathrm{SO}_0(1,n)$ we define the creation and annihilation operators by

$$a^\dagger(\vec{\zeta})\,|\Omega) = |\,\vec{\zeta}\,)\,, \qquad a(\vec{\zeta})\,|\Omega) = 0\,, \tag{6.21}$$

$$\Big[a(\vec{\zeta}),\, a^\dagger(\vec{\zeta}')\Big] = Q(\vec{\zeta}\cdot\vec{\zeta}')\,, \qquad [\,a(\vec{\zeta}),\, a(\vec{\zeta}')\,] = 0\,. \tag{6.22}$$

The field operator in the origin can be expanded in terms of these operators:

$$\Phi(0,\vec{\xi}_o) = \int d^{n-1}\Omega(\vec{\zeta})\ \Psi_o(\vec{\zeta})\,a^\dagger(\vec{\zeta}) \,+\, \Psi_o^*(\vec{\zeta})\,a(\vec{\zeta})\,, \tag{6.23}$$

where $d^{n-1}\Omega(\vec{\zeta})$ is the invariant volume element on S^{n-1}.

The covariance condition (6.20) determines the function $\Psi_o(\vec{\zeta}) = (\,\vec{\zeta}\,|\,\Phi(0,\,\vec{\xi}_o\,)\,\Omega)$. The rotation part of this equation implies that $\Psi_o(\vec{\zeta})$ depends on $\vec{\zeta}$ only through $\vec{\zeta}\cdot\vec{\xi}_o$, whereas the boost part fixes this dependence to be again governed by two arbitrary coefficients:

$$\Psi_o(\vec{\zeta}) = C\,\left(\vec{\zeta}\cdot\vec{\xi} - i\epsilon\right)^{-s-\frac{n-1}{2}} + D\,\left(\vec{\zeta}\cdot\vec{\xi} + i\epsilon\right)^{-s-\frac{n-1}{2}}\,. \tag{6.24}$$

Transporting the field with a boost followed by a rotation brings the point $(0,\vec{\xi}_o)$ to $(t,\vec{\xi})$. We thus get

$$\Phi(t,\,\vec{\xi}) \,=\, \int_{S^{n-1}} d^{n-1}\Omega(\vec{\zeta})\ \Psi_{t,\,\vec{\xi}}(\vec{\zeta})\,a^\dagger(\vec{\zeta}) \,+\, \Psi^*_{t,\,\vec{\xi}}(\vec{\zeta})\,a(\vec{\zeta})\,, \tag{6.25}$$

where $\Psi_{t,\,\vec{\xi}}(\vec{\zeta}) = (\,\vec{\zeta}\,|\,\Phi(t,\,\vec{\xi}\,)\,\Omega)$ is given by

$$\Psi_{t,\,\vec{\xi}}(\vec{\zeta}) = C\,\left(\cosh t\,\vec{\zeta}\cdot\vec{\xi} + \sinh t - i\epsilon\right)^{-s-\frac{n-1}{2}} + D\,\left(\cosh t\,\vec{\zeta}\cdot\vec{\xi} + \sinh t + i\epsilon\right)^{-s-\frac{n-1}{2}}\,. \tag{6.26}$$

At this point we are comparing with the results of the principal series I. The expression (6.26) is identical with that of the principal series with $s\to -i\mu$. The only difference is to be found in the commutation relations of creation and annihilation operators.

The two-point function is simply expressed in terms of $\Psi_{t,\xi}(\vec{\zeta})$ and the intertwiner $Q(\vec{\zeta}\cdot\vec{\zeta}')$ as

$$(\Omega|\,\Phi(t,\,\vec{\xi}\,)\,\Phi(t',\,\vec{\xi}'\,)\,\Omega) = \int_{S^{n-1}}\int_{S^{n-1}} d^{n-1}\Omega(\vec{\zeta})\,d^{n-1}\Omega(\vec{\zeta}')\ \Psi^*_{t,\,\vec{\xi}}(\vec{\zeta})\,Q(\vec{\zeta}\cdot\vec{\zeta}')\,\Psi_{t',\,\vec{\xi}'}(\vec{\zeta}')\,. \tag{6.27}$$

The above integral can be expressed in terms of hypergeometric functions as shown in the appendix B. We obtain

$$(\Omega|\,\Phi(x)\,\Phi(x')\,\Omega) = |C|^2\,F_n(x;x') + |D|^2\,F_n(\bar{x};\bar{x}') + 2\,\mathrm{Re}\Big[C^* D\,e^{i\pi\left(-s-\frac{n-1}{2}\right)}F_n(x;\bar{x}')\Big]\,, \tag{6.28}$$

with

$$F_n(x;x') = V_{n-1}\,\frac{s+\frac{n-1}{2}}{-s+\frac{n-1}{2}}\,{}_2F_1\left(s+\frac{n-1}{2}\,,\,-s+\frac{n-1}{2}\,;\,\frac{n}{2}\,;\,\frac{1+\tilde{Z}(x;x')}{2}\right), \tag{6.29}$$

where V_{n-1} is the area of $(n-1)$-sphere, $2\pi^{n/2}/\Gamma(n/2)$. We have defined $\tilde{Z}$ by

$$\tilde{Z}(x;x') = Z(x;x') + i\,\mathrm{sgn}(t-t')\,\epsilon\,, \tag{6.30}$$

where Z is the dS invariant quantity:

$$Z(t,\vec{\xi};t',\vec{\xi}') = \cosh t \cosh t' \, \vec{\xi}\cdot\vec{\xi}' - \sinh t \sinh t' = X^A X'_A \,. \tag{6.31}$$

The analysis of the equal time commutation relations made in I for the principal series applies also here and leads to commuting fields and

$$[\,\partial_t \Phi(0,\vec{\xi}_o\,),\, \Phi(0,\vec{\xi}\,)\,] = iN_n\, \delta^{n-1}(\,\vec{\xi}-\vec{\xi}_o\,)\,, \tag{6.32}$$

with N_n a constant which we shall shortly determine in terms of C and D. From the imaginary part of the short distance behavior of two-point function (6.28) which is given by

$$\begin{aligned} G_{dS_n}(x;x') \underset{x\to x'}{\approx} & \frac{2^{n-1}\,\pi^{\frac{n}{2}}\left(s+\frac{n-1}{2}\right)}{\Gamma\left(s+\frac{n-1}{2}\right)\Gamma\left(-s+\frac{n+1}{2}\right)} \times \\ & \times \begin{cases} |C|^2 \log\left\{(x-x')^2-(t-t'+i\epsilon)^2\right\}^{-2} + \\ \quad +|D|^2 \log\left\{(x-x')^2-(t-t'-i\epsilon)^2\right\}^{-2}\,, & \text{for } n=2\,, \\ \\ |C|^2 \left\{(\vec{x}-\vec{x}')^2-(t-t'+i\epsilon)^2\right\}^{\frac{2-n}{2}} + \\ \quad +|D|^2 \left\{(\vec{x}-\vec{x}')^2-(t-t'-i\epsilon)^2\right\}^{\frac{2-n}{2}}\,, & \text{for } n\neq 2\,, \end{cases} \end{aligned} \tag{6.33}$$

the requirement $N_n = 1$ leads to

$$|D|^2 - |C|^2 = \frac{\Gamma\left(s+\frac{n-1}{2}\right)\Gamma\left(-s+\frac{n+1}{2}\right)}{2^{n+1}\pi^n\left(s+\frac{n-1}{2}\right)}\,. \tag{6.34}$$

Another constraint on C and D can be obtained by demanding that the two point function coincides with the flat one in the small distance limit. This gives the so called Bunch-Davies vacuum [25]. In the coincidence point limit, the Minkowski vacuum positive Wightman function behaves as

$$G_{\text{Mink}}(x;x') \underset{x\to x'}{\approx} \frac{1}{4\pi^{\frac{n}{2}}} \times \begin{cases} \log\left\{(x-x')^2-(t-t'-i\epsilon)^2\right\}^{-2}\,, & \text{for } n=2\,, \\ \\ \Gamma(\frac{n}{2}-1)\left\{(\vec{x}-\vec{x}')^2-(t-t'-i\epsilon)^2\right\}^{\frac{2-n}{2}}\,, & \text{for } n\neq 2\,. \end{cases} \tag{6.35}$$

Imposing that the behavior be that of Hadamard, one gets the Bunch-Davies coefficients:

$$C^{\text{BD}} = 0\,, \qquad D^{\text{BD}} = \sqrt{\frac{\Gamma\left(s+\frac{n-1}{2}\right)\Gamma\left(-s+\frac{n+1}{2}\right)}{2^{n+1}\pi^n\left(s+\frac{n-1}{2}\right)}}\,. \tag{6.36}$$

The general coefficients C and D can be expressed in terms of the Bunch-Davies coefficients as

$$C = e^{i\beta}\sinh\alpha\, e^{-i\pi(s+\frac{n-1}{2})} D^{\text{BD}}\,, \qquad D = \cosh\alpha\, D^{\text{BD}}\,. \tag{6.37}$$

To study the asymptotic time behavior of the field operator and determine the α and β coefficients corresponding to the *in* and *out* vacua, it is useful to expand the field in the spherical harmonic basis as

$$\Phi(t,\,\vec{\xi}\,) \;=\; \sum_{L,k} c_L(t)\, S_{L,k}(\,\vec{\xi}\,)\, a^\dagger_{L,k} + \text{h.c.} \tag{6.38}$$

where the creation operators $a^\dagger_{L,k}$ are related to UIR states $|L,k)$. The coefficient $c_L(t)$ is calculated from eq. (6.26) by decomposing on the spherical harmonics as shown in appendix A, it reads

$$c_L(t) = \frac{2\left(\frac{2\pi}{\cosh t}\right)^{\frac{n-1}{2}}}{\Gamma(s+\frac{n-1}{2})}\Big[D\, e^{-i\pi(s+\frac{n-1}{2})}\left(\mathrm{Q}^s_{L+\frac{n-3}{2}}(-\tanh t)+i\,\frac{\pi}{2}\,\mathrm{P}^s_{L+\frac{n-3}{2}}(-\tanh t)\right) \tag{6.39}$$
$$+\,C\, e^{i\pi(s+\frac{n-1}{2})}\left(\mathrm{Q}^s_{L+\frac{n-3}{2}}(-\tanh t)-i\,\frac{\pi}{2}\,\mathrm{P}^s_{L+\frac{n-3}{2}}(-\tanh t)\right)\Big]\,,$$

where P^μ_ν and Q^μ_ν are associated Legendre functions of the first and second kind respectively. Using the parametrization (6.37), the asymptotic behavior in the remote past of $c_L(t)$ for non-integer s is shown in the appendix A to be given by

$$c_L(t) \underset{t\to-\infty}{=} P(e^{2t})\, e^{\left(s+\frac{n-1}{2}\right)t}\left\{1+i\,\omega^{\rm in}_L\,\left(2\,e^{-2st}\right)+o\left(e^{-2st}\right)\right\}, \tag{6.40}$$

where P is a real polynomial of degree less than $-s$ up to an overall phase and the frequency $\omega^{\rm in}_L$ is given by

$$\omega^{\rm in}_L = \frac{\Gamma(1+s)\,\Gamma(L-s+\frac{n-1}{2})}{2\,\Gamma(1-s)\,\Gamma(L+s+\frac{n-1}{2})}\,\frac{-\sin\pi s+i(\cos\pi s\cosh 2\alpha+\cos(\beta+\frac{n-1}{2}\pi)\sinh 2\alpha)}{\cosh 2\alpha+\cos\left(\beta+\left(s+\frac{n-1}{2}\right)\pi\right)\sinh 2\alpha}\,. \tag{6.41}$$

Factorizing the overall decreasing term and defining a new time coordinate as eq. (3.42), the last factor of $c_L(t)$ can be written as a plane wave to the first order in η_s as the two-dimensional case (3.43). The *in* vacuum with respect to η_s is defined by $\omega^{\rm in}_L$ real and positive. Since the real part of $\omega^{\rm in}_L$ is always positive, the condition $\mathrm{Im}[\omega^{\rm in}_L]=0$ is sufficient and reads

$$\cos\pi s = -\tanh 2\alpha\,\cos\left(\beta+\frac{n-1}{2}\pi\right)\,. \tag{6.42}$$

This condition interpolates between the BD vacuum in the conformally massless case which in n-dimensions corresponds to $s=-1/2$ and the Mottola-Schwinger *in* vacuum of the principal series which is given with the conditions [15]:

$$\cosh\pi\mu = \coth 2\alpha\,,\qquad \cos\left(\beta+\frac{n-1}{2}\pi\right) = -1\,, \tag{6.43}$$

where $\mu = i\,s\in\mathbb{R}$.

The large future limit is similarly determined in the appendix to be given by

$$c_L(t) \underset{t\to\infty}{=} Q(e^{-2t})\, e^{-\left(s+\frac{n-1}{2}\right)t}\left\{1+i\,\omega^{\rm out}_L\,\left(-2\,e^{2st}\right)+o\left(e^{2st}\right)\right\}, \tag{6.44}$$

where Q is a real polynomial of degree less than $-s$ multiplied by an overall constant and where $\omega^{\rm out}_L$ is given by

$$\omega^{\rm out}_L = \frac{\Gamma(1+s)\,\Gamma(L-s+\frac{n-1}{2})}{2\,\Gamma(1-s)\,\Gamma(L+s+\frac{n-1}{2})}\,\frac{-\sin\pi s+i(\cos\pi s\cosh 2\alpha+\cos(\beta-\frac{n-1}{2}\pi)\sinh 2\alpha)}{\cosh 2\alpha+\cos\left(\beta-\left(s+\frac{n-1}{2}\right)\pi\right)\sinh 2\alpha}\,. \tag{6.45}$$

Using a time coordinate η_s given in eq. (3.48), the last factor of $c_L(t)$ becomes a plane wave to the first order in η_s and the reality of $\omega_L^{\rm out}$ defines the *out* vacuum:

$$\cos\pi s = -\tanh 2\alpha\,\cos\left(\beta - \frac{n-1}{2}\pi\right)\,. \tag{6.46}$$

The resulting $\omega_L^{\rm out}$ is positive. Comparing with the *out* vacuum in the principal series:

$$\cosh\pi\mu = \coth 2\alpha\,, \qquad \cos\left(\beta - \frac{n-1}{2}\pi\right) = -1\,, \tag{6.47}$$

we can see that this newly defined *out* vacuum also interpolates between the BD vacuum in the conformally massless case $s=-1/2$ and the Mottola-Schwinger *out* vacuum in the principal series.

Notice that contrary to the case of the principal series we get a family of dS invariant *in* and *out* vacua parametrized by β. For the time-reversal *in* and *out* vacua with $\alpha^{\rm out}=\alpha^{\rm in}$, $\beta^{\rm out}=-\beta^{\rm in}$, the mean number of *out* quanta of momentum L in *in* vacuum can be easily obtained as

$$\bar n_{\rm out/in} = \cosh^2(2\alpha^{\rm in})\,\cos^2\pi s = \frac{\cos^2\left(\beta^{\rm in}-\frac{n-1}{2}\pi\right)\,\cos^2\pi s}{\cos^2\left(\beta^{\rm in}-\frac{n-1}{2}\pi\right)-\cos^2\pi s}\,. \tag{6.48}$$

It vanishes, as it should, in the conformal case $s=-1/2$ and diverges in the $s\to 0$ limit. Notice that the number also vanishes for s half integer and tends to infinity when s approaches an integer. It should also be noticed that for n odd the family of *in* vacua coincides with the family of *out* vacua, this is to be compared with the fact that in odd dimensions the *in* and *out* vacua also coincide for the principal series. Notice that when $\beta^{\rm in}=(n-1)\pi/2$, the number of created quanta is minimal and is given by $\cot^2\pi s$. It is easy to show that this is also the minimal number of created quanta when considering all *in* and *out* vacua.

It remains to examine the case of integer s, in this case the asymptotic behavior of $c_L(t)$ in the remote past is given by

$$c_L(t) \underset{t\to-\infty}{=} R(e^{2t})\,e^{\left(s+\frac{n-1}{2}\right)t}\left\{1+i\,\nu_L\,\left(2\,e^{-2st}\right)+o\left(e^{-2st}\right)\right\}, \tag{6.49}$$

where R is a decreasing real function with overall complex constant and the frequency ν_L is given by

$$\nu_L = \frac{\Gamma(L+\frac{n-1}{2}-s)}{2\,\Gamma(-s)\,\Gamma(1-s)\,\Gamma(L+\frac{n-1}{2}+s)}\left[\pi\,\frac{1-i\sinh 2\alpha\sin(\beta+s+\frac{n-1}{2})}{\cosh 2\alpha+\cos(\beta+s+\frac{n-1}{2})\sinh 2\alpha}+ \right. \tag{6.50}$$
$$\left. -i\Big\{\psi(1)+\psi(1-s)-\psi(L+\tfrac{n-1}{2})-\psi(L+\tfrac{n-1}{2}-s)\Big\}\right],$$

where ψ is digamma function. Requiring ν_L to be real, we obtain a constraint on the value of α and β as in the case of non-integer s. However in the present case, the condition that α and β should satisfy depends also on the mode L, which means thus defined *in* vacuum is not dS invariant.

6.4 Massless limit of massive field

We consider the massless limit of the massive scalar field from the complementary series. As $\varepsilon = s + (n-1)/2$ approaches to zero, the zero mode coefficient, $c_0(t)$, in eq. (6.39) diverges, since the associated Legendre function of the second kind $\mathrm{Q}_{L+\frac{n-3}{2}}^{\varepsilon-\frac{n-1}{2}}$ diverges when ε approaches zero.

This divergence can be also more explicitly seen in the position basis. If we expand $\Psi_{\eta,\vec{\xi}}$ given in eq. (6.26) in ε, using eq. (6.36) and eq. (6.37), we get

$$\begin{aligned}\Psi_{\eta,\vec{\xi}}(\vec{\zeta}) = & \sqrt{\frac{\Gamma(n)}{2^{n+1}\pi^n}} \times \\ & \times \left[\left(\cosh\alpha + e^{i\beta}\sinh\alpha\right) \left(\frac{1}{\varepsilon} + \ln(\sin\eta) - \ln\left|\vec{\zeta}\cdot\vec{\xi} - \cos\eta\right| + \frac{1}{2}(\psi(1)-\psi(n)) - i\frac{\pi}{2}\right) + \right. \\ & \left. + i\,\pi \left(\cosh\alpha - e^{i\beta}\sinh\alpha\right) \left\{\frac{1}{2} - \Theta\left(\cos\eta - \vec{\zeta}\cdot\vec{\xi}\right)\right\}\right] + \mathcal{O}(\varepsilon). \end{aligned} \tag{6.51}$$

The only divergent term in the $\varepsilon \to 0$ limit is a constant, i.e. it contributes only to the zero mode $c_0(\eta)$. Using eq. (6.51), we get the small ε behavior of the zero mode as

$$\begin{aligned} c_0(\eta) = & \sqrt{\frac{V_{n-1}}{V_n}} \left[\left(\cosh\alpha + e^{i\beta}\sinh\alpha\right) \left(\frac{1}{\varepsilon} + f(\eta) - i\frac{\pi}{2}\right) \right. \\ & \left. + i\left(\cosh\alpha - e^{i\beta}\sinh\alpha\right) g(\eta) \right] + \mathcal{O}(\varepsilon), \end{aligned} \tag{6.52}$$

where f and g are defined by

$$\begin{aligned} f(\eta) &= \ln(\sin\eta) + \frac{1}{2}\left(\psi(1) - \psi(n)\right) - \frac{V_{n-2}}{V_{n-1}} \int_0^\pi d\phi\, \sin^{n-2}\phi\, \ln|\cos\eta - \cos\phi|, \\ g(\eta) &= \frac{\pi\, V_{n-2}}{V_{n-1}} \int_{\frac{\pi}{2}}^{\eta} d\phi\, \sin^{n-2}\phi = -\frac{\pi\, V_{n-2}}{V_{n-1}} \cos\eta\; {}_2F_1\left(\frac{1}{2}, \frac{3-n}{2}; \frac{1}{2}; \cos^2\eta\right). \end{aligned} \tag{6.53}$$

Another source of singularity in the $\varepsilon \to 0$ limit is the vanishing of the norm of the zero mode. This translates into the vanishing of the commutator:

$$[\,a_0\,,\, a_0^\dagger\,] = \frac{\varepsilon}{n-1-\varepsilon}\,. \tag{6.54}$$

The combination $(c_0\, a_0^\dagger + c_0^*\, a_0)/\sqrt{V_{n-1}}$ which appears in the zero mode part of the field operator can be put in the form

$$q_{\alpha,\beta} + p_{\alpha,\beta}\, \frac{g(\eta)}{\pi\, V_{n-2}} \tag{6.55}$$

where $q_{\alpha,\beta}$ and $p_{\alpha,\beta}$ are given by

$$\begin{aligned} q_{\alpha,\beta} &= \frac{1}{\varepsilon\sqrt{V_n}} \left\{ \left(\cosh\alpha + e^{i\beta}\sinh\alpha\right) a_0^\dagger + \text{h.c.} \right\}, \\ p_{\alpha,\beta} &= i\sqrt{V_n}\left(\frac{n-1-\varepsilon}{2}\right) \left\{ \left(\cosh\alpha - e^{i\beta}\sinh\alpha\right) a_0^\dagger - \text{h.c.} \right\}, \end{aligned} \tag{6.56}$$

and they obey canonical commutation relations $[\, q_{\alpha,\beta}\, ,\, p_{\alpha,\beta}\,] = i\,$. Finally, the scalar field in the massless limit is expressed as

$$\Phi(\eta,\vec{\xi}) \;=\; q_{\alpha,\beta} \,+\, p_{\alpha,\beta}\, \frac{g(\eta)}{\pi\, V_{n-2}} \,+ \int_{S^{n-1}} d^{n-1}\Omega(\vec{\zeta})\, \Upsilon_{\eta,\vec{\xi}}(\vec{\zeta})\, a^\dagger_*(\vec{\zeta}) + \text{h.c.} \tag{6.57}$$

where the non-zero part of creation operator $a^\dagger_*(\vec{\zeta})$ and its coefficient $\Upsilon_{\eta,\vec{\xi}}$ are defined by

$$a^\dagger_*(\vec{\zeta}) \equiv a^\dagger(\vec{\zeta}) - \frac{1}{\sqrt{V_{n-1}}}\, a^\dagger_0\,, \qquad [\, a_*(\,\vec{\zeta}\,)\,,\, a^\dagger_*(\,\vec{\zeta}'\,)\,] = Q(\vec{\zeta}\cdot\vec{\zeta}')\,, \tag{6.58}$$

$$\Upsilon_{t,\vec{\xi}}(\vec{\zeta}) \;=\; -\sqrt{\frac{\Gamma(n)}{2^{n+1}\pi^n}}\, \Big[e^{i\beta} \sinh\alpha\, \ln\Big(\cosh t\vec{\zeta}\cdot\vec{\xi} - \sinh t - i\epsilon\Big) + \\ + \cosh\alpha\, \ln\Big(\cosh t\vec{\zeta}\cdot\vec{\xi} - \sinh t + i\epsilon\Big) \Big]\,.$$

The above expression has a well defined limit.

As in the two-dimensional case, as ε goes to zero with finite $q_{\alpha,\beta}$ and $p_{\alpha,\beta}$, the defining condition of the zero mode vacuum state reduces to $p_{\alpha,\beta}\,|\Omega) = 0$. Such a state cannot be normalizable and the resulting Fock space is therefore the tensor product of the Fock space constructed from massless representation and the Hilbert space carrying a representation of $[\, q_{\alpha,\beta}\, ,\, p_{\alpha,\beta}\,] = i\,$, i.e. the one describing a one dimensional quantum mechanical particle.

The resulting generators of the dS group are deformed by $p_{\alpha,\beta}$: in the complementary series the zero mode part of M_{I0} can be written as

$$M^{(0)}_{I0} = \sum_{J=1}^{n} (1, J|\, M_{I0}\,|\, 0)\, \frac{1}{Q_0}\, a^\dagger_{1,J}\, a_0 \,+\, (0\,|\, M_{I0}\,|1, J)\, \frac{1}{Q_1}\, a^\dagger_0\, a_{1,J}\,, \tag{6.59}$$

where $|\,0)$ and $|1, J)$ are the states in the spherical harmonic basis:

$$(\,\vec{\zeta}\,|\, 0) = S_0(\,\vec{\zeta}\,) = \frac{1}{\sqrt{V_{n-1}}}\,, \qquad (\,\vec{\zeta}\,|\, 1, J) = S_{1,J}(\,\vec{\zeta}\,) = \sqrt{\frac{n}{V_{n-1}}}\, \zeta^J\,, \tag{6.60}$$

In terms of the redefined zero mode operators $p_{\alpha,\beta}$ and $q_{\alpha,\beta}$, eq. (6.59) has a limit given by

$$M^{(0)}_{I0} = \frac{1}{\sqrt{n\, V_n}}\, \frac{p_{\alpha,\beta}\Big(a^\dagger_{1,I} + a_{1,I}\Big)}{\cosh\alpha + \cos\beta\sinh\alpha}\,. \tag{6.61}$$

This part should be added to the generators obtained from the massless UIR of the dS group. The dS invariance of the vacuum state leads again $p_{\alpha,\beta}\,|\Omega) = 0\,$, making the vacuum state non-normalizable.

6.5 Vertex operator

In order to make the vacuum state normalizable, we compactify the scalar field on a circle of radius L as in the two-dimensional case. The physical observables should then be invariant under $\Phi \to \Phi + 2\pi L$. This is the case for the vertex operator V which is a dS covariant regularization of $\exp(i\Phi/L)\,$.

As in two dimensional case, this regularization can be realized by defining V at the origin as the normal ordered exponential (5.16) and transporting with the dS transformations. The resulting vertex operator can be expressed as

$$V(\eta,\vec{\xi}) = \exp\left(\frac{i}{L}\phi^{+}(\eta,\vec{\xi})\right)\exp\left(\frac{i}{L}\phi^{-}(\eta,\vec{\xi})\right)\exp\left(\frac{i}{L}\phi^{0}(\eta,\vec{\xi})\right)\times$$
$$\times\exp\left(\frac{1}{2V_{n-1}L^2}\lim_{\varepsilon\to 0}\left(|c_0(\eta)|^2-|c_0(\tfrac{\pi}{2})|^2\right)[\,a_0\,,\,a_0^{\dagger}\,]\right), \tag{6.62}$$

where ϕ^{+}, ϕ^{-} and ϕ^{0} are respectively creation, annihilation and zero mode part of scalar field operator. Notice that it differs from the normal ordered exponential which is not dS invariant. The difference is a time dependent constant given in the last factor in eq. (6.62) and it reads explicitly

$$\exp\left[\frac{(\cosh 2\alpha+\sinh 2\alpha\cos\beta)\,(f(\eta)-f(\frac{\pi}{2}))+\sinh 2\alpha\sin\beta\,(g(\eta)-g(\frac{\pi}{2}))}{2\pi\,V_{n-2}\,L^2}\right]. \tag{6.63}$$

Two-point function of the vertex operators $(\Omega|\,V^{\dagger}(x)\,V(x')\,\Omega)\equiv\exp\left(\frac{1}{L^2}\,\mathcal{G}_n(x;x')\right)$ can now be easily calculated and expressed in terms of the limit of the massive two-point function as

$$\mathcal{G}_n(x;x') = \lim_{\varepsilon\to 0}\left\{(\Omega|\,\Phi_\varepsilon(x)\,\Phi_\varepsilon(x')\,\Omega) - \frac{|c_0(t=0)|^2}{V_{n-1}}[\,a_0\,,\,a_0^{\dagger}\,]\right\}. \tag{6.64}$$

Using the two-point function (6.28) and the zero mode (6.52), the divergent part cancels and we get a finite limit

$$2\pi\,V_{n-2}\,\mathcal{G}_n(x;x') = \sinh^2\alpha\,G_n(x;x')+\cosh^2\alpha\,G_n(\bar{x};\bar{x}')$$
$$+\,2\,\mathrm{Re}\left[\sinh\alpha\cosh\alpha\,e^{-i\beta}\,G_n(x;\bar{x}')\right], \tag{6.65}$$

where the dS invariant function $G_n(x;x')$ is defined by

$$G_n(x;x') = \frac{\partial}{\partial\varepsilon}\,{}_2F_1\left(-\varepsilon+n-1\,,\,\varepsilon\,;\,\frac{n}{2}\,;\,\frac{1+\tilde{Z}(x;x')}{2}\right)\Bigg|_{\varepsilon=0}+\psi\left(\frac{n-1}{2}\right)-\psi\left(\frac{n}{2}\right). \tag{6.66}$$

Notice that the procedure we propose results in the extraction from the massive two point function of its divergent part. In fact, $(\Omega|\,\Phi_\varepsilon(x)\,\Phi_\varepsilon(x')\,\Omega)$ behaves in the massless limit as

$$\frac{\cosh 2\alpha+\sinh 2\alpha\cos\beta}{2\pi\,V_{n-2}}\left(\frac{1}{\varepsilon}+\psi(1)-\psi(n-1)-\psi\left(\frac{n-1}{2}\right)+\psi\left(\frac{n}{2}\right)\right)+\mathcal{G}_n(x;x'). \tag{6.67}$$

The function $G_n(x;x') = f(Z(x;x'))$ can be determined from the differential equation satisfied by ${}_2F_1(-\varepsilon+n-1\,,\,\varepsilon\,;\,n/2\,;\,(1+\tilde{Z}(x;x'))/2)$ which results in

$$(1-x^2)\,f''(x)-n\,x\,f'(x)-(n-1)=0, \tag{6.68}$$

and its explicit expression is given in appendix C.

An interesting and unexpected property of $\mathcal{G}_n$ is its behavior for large $|Z|$, that is in the far infrared region in the flat sections. In this limit eq. (6.68) simplifies to $(x^n f')' = -(n-1)\,x^{n-2}$ and its solution for large x is $-\log x$ independently of n. The *massless two-point function* has thus a logarithmic divergence for all spacetime dimensions similar to the two dimensional flat infrared divergence. This IR divergence was found in [24] by considering the BD massless two-point function and disregarding its divergent part. Here, this regularization arises naturally in a dS invariant way and is due to the compactification of the scalar on a circle. In this respect, this IR divergence was argued to cause a restoration of a breakdown of symmetry [24] similarly to what happens in two dimensions (For discussions on IR divergence for the graviton, see e.g. [28]). The two-point function of the vertex operators can be used to exhibit this symmetry restoration. From our previous analysis for x and x' separated by a large spacelike distance $d(x;x') = \cosh^{-1} Z(x;x')$, we have

$$(\Omega|\, V^\dagger(x)\, V(x')\,\Omega) \sim Z(x;x')^{-\frac{\cosh 2\alpha + \sinh 2\alpha \cos\beta}{2\pi\, V_{n-2}\, L^2}}\,, \tag{6.69}$$

which tends to zero. This is a signal of large quantum fluctuations which restore a broken symmetry.

A. Intertwiner and Field coefficients in spherical harmonic basis

In this subsection, we derive the intertwiner Q_L and the field coefficient $c_L(t)$ in the spherical harmonic basis. Many properties of special functions used in the following can be found in [29, 30]

At first, we concentrate on the case of the intertwiner operator. From the n-dimensional addition theorem which reads

$$\sum_{k=1}^{N(n,L)} (\,\vec\xi\,|L,k)(L,k|\,\vec\zeta\,) = \sum_{k=1}^{N(n,L)} S_{L,k}(\vec\xi) S_{L,k}(\vec\zeta) = \frac{2L+n-2}{(n-2)V_{n-1}}\, C_L^{\frac{n-2}{2}}(\vec\zeta\cdot\vec\xi)\,, \tag{A.1}$$

with $C_L^{\frac{n-2}{2}}$ the Gegenbauer polynomial, we get the relation between the intertwiners in the position basis and in the spherical harmonic basis:

$$Q(\vec\zeta\cdot\vec\zeta') = \sum_{L,k} Q_L\, S_{L,k}(\vec\zeta)\, S_{L,k}(\vec\zeta') = \sum_L Q_L\, \frac{2L+n-2}{(n-2)V_{n-1}}\, C_L^{\frac{n-2}{2}}(\vec\zeta\cdot\vec\zeta')\,. \tag{A.2}$$

Using the orthogonality of the Gegenbauer polynomials, we express Q_L as

$$Q_L = (4\pi)^{\frac{n-2}{2}} \frac{\Gamma(L+1)\,\Gamma(\frac{n-2}{2})}{\Gamma(L+n-2)} \int_{-1}^{1} dx\,(1-x^2)^{\frac{n-3}{2}} C_L^{\frac{n-2}{2}}(x)\, Q(x)\,. \tag{A.3}$$

On the other hand, the Gegenbauer polynomials are given by

$$C_L^{\frac{n-2}{2}}(x) = \frac{(-1)^L\,\pi^{\frac12}\,2^{-L-n+3}\,\Gamma(L+n-2)}{\Gamma(L+\frac{n-1}{2})\,\Gamma(L+1)\,\Gamma(\frac{n-2}{2})}(1-x^2)^{-\frac{n-3}{2}}\left(\frac{d}{dx}\right)^L (1-x^2)^{L+\frac{n-3}{2}}\,. \tag{A.4}$$

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Using the expression of $Q(\vec{\zeta}{\cdot}\vec{\zeta}')$ in eq. (6.8) and applying an integration by part, we get

$$Q_L = Q_0 \, \frac{\Gamma(s+\frac{n-1}{2})\,\Gamma(-s+\frac{n-1}{2}+L)}{2^{s+\frac{n-3}{2}+L}\Gamma(s)\,\Gamma(L+\frac{n-1}{2})\,\Gamma(-s+\frac{n-1}{2})} \int_{-1}^{1} dx\,(1-x)^{s-\frac{n-1}{2}-L}(1-x^2)^{L+\frac{n-3}{2}}\,. \tag{A.5}$$

Performing the integral we get an expression of Q_L which coincides eq. (6.11).

The coefficient of field operator $c_L(t)$ can also be obtained in a similar manner. We first decompose $\Psi_{t,\,\vec{\xi}}(\vec{\zeta})$ in the Gegenbauer polynomials as

$$\begin{aligned}\Psi_{t,\vec{\zeta}}(\vec{\xi}) &= \sum_{L,k} (\,\vec{\zeta}\,|L,k)(L,k|\,\Phi(t,\,\vec{\xi}\,)\,\Omega) = \sum_{L,k} c_L(t)\, S_{L,k}(\vec{\zeta})\, S_{L,k}(\vec{\xi}) \\ &= \sum_{L} c_L(t)\,\frac{2L+n-2}{(n-2)V_{n-1}}\, C_L^{\frac{n-2}{2}}(\vec{\zeta}{\cdot}\vec{\xi})\,,\end{aligned} \tag{A.6}$$

and using their orthogonality, we express $c_L(t)$ as an integral:

$$c_L(t) = (4\pi)^{\frac{n-2}{2}} \frac{\Gamma(L+1)\,\Gamma(\frac{n-2}{2})}{\Gamma(L+n-2)} \int_{-1}^{1} dx\,(1-x^2)^{\frac{n-3}{2}} C_L^{\frac{n-2}{2}}(x)\,\Psi(t,x)\,, \tag{A.7}$$

where $\Psi(t,x) = C\,(\cosh t\, x+\sinh t - i\epsilon)^{-s-\frac{n-1}{2}} + D\,(\cosh t\, x+\sinh t + i\epsilon)^{-s-\frac{n-1}{2}}$. This integral can be again evaluated easily using eq. (A.4) and the integral representation of the hypergeometric function. Finally we get

$$\begin{aligned} c_L(t) = {} & \frac{\pi^{\frac{n+2}{2}}}{\Gamma\left(L+\frac{n}{2}\right)} \left(\frac{\cosh t}{2}\right)^{L} e^{\left(s+\frac{n-1}{2}+L\right)t} \,\times \\ & \times \left[D\, e^{-i\pi\left(s+\frac{n-1}{2}\right)} {}_2F_1\left(L+\frac{n-1}{2}\,,\, L+\frac{n-1}{2}+s\,;\, 2L+n-1\,;\, 1+e^{2t}+i\epsilon\right) + \right. \\ & \left. + C\, e^{i\pi\left(s+\frac{n-1}{2}\right)} {}_2F_1\left(L+\frac{n-1}{2}\,,\, L+\frac{n-1}{2}+s\,;\, 2L+n-1\,;\, 1+e^{2t}-i\epsilon\right) \right]. \end{aligned} \tag{A.8}$$

Since the hypergeometric functions ${}_2F_1(a,b;c;z)$ have a branch cut on $\mathrm{Arg}[z-1]=0$, the above two hypergeometric functions with different $i\epsilon$ prescriptions are independent. Using the transformation identities between the hypergeometric and the associated Legendre functions, the above expression of $c_L(t)$ can be written also as eq. (6.39).

In the subsection 6.3, we used the asymptotic behavior of $c_L(t)$ in order to define *in* and *out* vacua. These asymptotic behaviors can be obtained from those of the hypergeometric function. We concentrate on the large past case, $t\to-\infty$ and the large future case can be done in a similar manner. When s is not a integer and in the interval $]-m-1,-m[$, the asymptotic behavior of the hypergeometric function is given by

$$
\begin{aligned}
&{}_2F_1\left(L+\frac{n-1}{2}\,,\,L+\frac{n-1}{2}+s\,;\,2L+n-1\,;\,1+e^{2t}-i\epsilon\right)\\
&\quad=\frac{\Gamma(2L+n-1)\,\Gamma(-s)}{\Gamma(L+\frac{n-1}{2})\,\Gamma(L+\frac{n-1}{2}-s)}\left(\sum_{l=0}^{m}\frac{(L+\frac{n_1}{2})_{(l)}(L+\frac{n-1}{2}+s)_{(l)}}{(1+s)_{(l)}\,l!}(-1)^l e^{2lt}+\right.\\
&\qquad\qquad\left.+\frac{\Gamma(s)\,\Gamma(L+\frac{n-1}{2}-s)}{\Gamma(-s)\,\Gamma(L+\frac{n-1}{2}+s)}e^{-i\pi s}e^{-2st}+o(e^{-2st})\right)\\
&\quad=p(e^{2t})\left(1+\frac{\Gamma(s)\,\Gamma(L+\frac{n-1}{2}-s)}{\Gamma(-s)\,\Gamma(L+\frac{n-1}{2}+s)}e^{-i\pi s}e^{-2st}+o(e^{-2st})\right),
\end{aligned}
\tag{A.9}
$$

where $p(e^{2t})$ is a real polynomial of order m in e^{2t} and given by

$$
p(e^{2t})=\frac{\Gamma(2L+n-1)\,\Gamma(-s)}{\Gamma(L+\frac{n-1}{2})\,\Gamma(L+\frac{n-1}{2}-s)}\sum_{l=0}^{m}\frac{(L+\frac{n-1}{2})_{(l)}(L+\frac{n-1}{2}+s)_{(l)}}{(1+s)_{(l)}\,l!}(-1)^l e^{2lt}\,. \tag{A.10}
$$

Combining the two hypergeometric functions with coefficients C and D, we get the asymptotic behavior of $c_L(t)$ as

$$
\begin{aligned}
c_L(t)&=\frac{\pi^{\frac{n+2}{2}}}{4^L\Gamma\left(L+\frac{n}{2}\right)}\left(D\,e^{-i\pi\left(s+\frac{n-1}{2}\right)}+C\,e^{i\pi\left(s+\frac{n-1}{2}\right)}\right)p(e^{2t})\,e^{\left(s+\frac{n-1}{2}\right)t}\times\\
&\times\left(1+\frac{D\,e^{-i\pi\frac{n-1}{2}}+C\,e^{i\pi\frac{n-1}{2}}}{D\,e^{-i\pi\left(s+\frac{n-1}{2}\right)}+C\,e^{i\pi\left(s+\frac{n-1}{2}\right)}}\frac{\Gamma(s)\,\Gamma(L+\frac{n-1}{2}-s)}{\Gamma(-s)\,\Gamma(L+\frac{n-1}{2}+s)}e^{-i\pi s}e^{-2st}+o(e^{-2st})\right),
\end{aligned}
\tag{A.11}
$$

this gives eq. (6.40) after using eq. (6.41) and defining $P(e^{2t})$ as $p(e^{2t})$ times the overall constant of $c_L(t)$ in eq. (A.9).

For the case of integer s, the expression (A.9) is replaced by

$$
\begin{aligned}
&\frac{\Gamma(2L+n-1)}{\Gamma(L+\frac{n-1}{2})}\left[\frac{\Gamma(-s)}{\Gamma(L+\frac{n-1}{2}-s)}\sum_{l=0}^{-s-1}\frac{(L+\frac{n-1}{2})_{(l)}(L+\frac{n-1}{2}+s)_{(l)}}{(1+s)_{(l)}\,l!}(-1)^l e^{2lt}\right.\\
&\qquad\left.-\frac{e^{-2st}}{\Gamma(L+\frac{n-1}{2}+s)\,\Gamma(1-s)}\left\{2t+i\pi-\psi(1)-\psi(1-s)+\psi(L+\tfrac{n-1}{2})+\psi(L+\tfrac{n-1}{2}-s)\right\}\right]\\
&\qquad=r(e^{2t})\left(1-\frac{\Gamma(L+\frac{n-1}{2}-s)}{\Gamma(L+\frac{n-1}{2}+s)}\frac{e^{-2st}}{\Gamma(-s)\,\Gamma(1-s)}\times\right.\\
&\qquad\qquad\left.\times\left\{i\pi-\psi(1)-\psi(1-s)+\psi(L+\tfrac{n-1}{2})+\psi(L+\tfrac{n-1}{2}-s)\right\}\right),
\end{aligned}
\tag{A.12}
$$

where $r(e^{2t})$ is given by

$$
\begin{aligned}
r(e^{2t})=\frac{\Gamma(2L+n-1)}{\Gamma(L+\frac{n-1}{2})}\frac{\Gamma(-s)}{\Gamma(L+\frac{n-1}{2}-s)}&\left(\sum_{l=0}^{-s-1}\frac{(L+\frac{n-1}{2})_{(l)}(L+\frac{n-1}{2}+s)_{(l)}}{(1+s)_{(l)}\,l!}(-1)^l e^{2lt}\right.\\
&\left.-\frac{2\,\Gamma(L+\frac{n-1}{2}-s)}{\Gamma(L+\frac{n-1}{2}+s)\,\Gamma(-s)\,\Gamma(1-s)}\,e^{-2st}\,t\right).
\end{aligned}
$$

Combining again two hypergeometric functions with C and D, the asymptotic behavior of $c_L(t)$ with integer value of s is given by eq. (6.49), where $R(e^{2t})$ is $r(e^{2t})$ times the overall constant in eq. (A.9).

B. Wightman function

The two-point function in n-dimensions given in eq. (6.27) reads

$$(\Omega|\,\Phi(t,\,\vec{\xi}\,)\,\Phi(t',\,\vec{\xi}\,'\,)\,\Omega) \,=\, Q_0\,\frac{\Gamma\left(s+\frac{n-1}{2}\right)}{(2\pi)^{\frac{n-1}{2}}\,2^s\,\Gamma(s)}\,\times \\ \times\int_{S^{n-1}\times S^{n-1}} d^{n-1}\Omega(\,\vec{\zeta}\,)\,d^{n-1}\Omega(\,\vec{\zeta}\,')\,\left(1-\vec{\zeta}\cdot\vec{\zeta}\,'\right)^{s-\frac{n-1}{2}}\,\Psi^*_{t,\,\vec{\xi}}(\,\vec{\zeta}\,)\,\Psi_{t',\,\vec{\xi}\,'}(\,\vec{\zeta}\,'\,)\,, \tag{B.1}$$

with $\Psi_{t,\,\vec{\xi}}(\,\vec{\zeta}\,)$ given before. Each term with coefficients $|C|^2$, $|D|^2$, C^*D and D^*C can be written in terms of a single function F_n with appropriate arguments as

$$(\Omega|\,\Phi(t,\,\vec{\xi}\,)\,\Phi(t',\,\vec{\xi}\,'\,)\,\Omega) \,=\, |C|^2\,F_n(x;x')+|D|^2\,F_n(\bar{x};\bar{x}')+ \\ +C^*D\,e^{i\pi\left(-s-\frac{n-1}{2}\right)}\,F_n(x;\bar{x}')\,+\,D^*C\,e^{-i\pi\left(-s-\frac{n-1}{2}\right)}\,F_n(\bar{x};x')\,, \tag{B.2}$$

with

$$F_n(t,\,\vec{\xi};t',\,\vec{\xi}\,') \,=\, Q_0\,\frac{\Gamma\left(s+\frac{n-1}{2}\right)}{2^s\,(2\pi)^{\frac{n-1}{2}}\,\Gamma(s)}\,\times \\ \times\int_{S^{n-1}}\int_{S^{n-1}} d^{n-1}\Omega(\vec{\zeta})\,d^{n-1}\Omega(\vec{\zeta}')\,\left(1-\vec{\zeta}\cdot\vec{\zeta}'\right)^{s-\frac{n-1}{2}}\,\times \\ \times\left(\cosh t\,\vec{\zeta}\cdot\vec{\xi}+\sinh t+i\epsilon\right)^{-s-\frac{n-1}{2}}\,\left(\cosh t'\,\vec{\zeta}'\cdot\vec{\xi}'+\sinh t'-i\epsilon\right)^{-s-\frac{n-1}{2}}\,. \tag{B.3}$$

Using the following integral obtained in I:

$$\int d^{n-1}\Omega(\vec{\zeta})\,\left(\cosh t\,\vec{\zeta}\cdot\vec{\xi}+\sinh t+i\epsilon\right)^{s-\frac{n-1}{2}}\,\left(\cosh t'\,\vec{\zeta}\cdot\vec{\xi}'+\sinh t'-i\epsilon\right)^{-s-\frac{n-1}{2}} \\ =\,e^{i\pi s}\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\,{}_2F_1\left(i\mu+\frac{n-1}{2},-i\mu+\frac{n-1}{2};\frac{n}{2};\frac{1+\tilde{Z}(t,\vec{\xi};t',\vec{\xi}')}{2}\right)\,, \tag{B.4}$$

where

$$\tilde{Z}(t,\vec{\xi};t',\vec{\xi}') = \cosh t\cosh t'\,\vec{\xi}\cdot\vec{\xi}' - \sinh t\sinh t' + i\,\mathrm{sgn}(t-t')\,\epsilon\,, \tag{B.5}$$

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the integral with respect to $\vec{\zeta}'$ can be expressed as a limit of hypergeometric function:

$$\begin{aligned}
&\int_{S^{n-1}} d^{n-1}\Omega(\vec{\zeta}')\,\left(1-\vec{\zeta}\cdot\vec{\zeta}'\right)^{s-\frac{n-1}{2}}\left(\cosh t'\,\vec{\zeta}'\cdot\vec{\xi}'+\sinh t'-i\epsilon\right)^{-s-\frac{n-1}{2}}\\
&=(e^{-i\pi})^{s-\frac{n-1}{2}}\lim_{t\to-\infty}\left\{(2e^t)^{s-\frac{n-1}{2}}\int_{S^{n-1}} d^{n-1}\Omega(\vec{\zeta}')\,\left(\cosh t\,\vec{\zeta}\cdot\vec{\zeta}'+\sinh t+i\epsilon\right)^{s-\frac{n-1}{2}}\times\right.\\
&\qquad\qquad\left.\times\left(\cosh t'\,\vec{\zeta}'\cdot\vec{\xi}'+\sinh t'-i\epsilon\right)^{-s-\frac{n-1}{2}}\right\}\\
&=(e^{-i\pi})^{s-\frac{n-1}{2}}\lim_{t\to-\infty}\left\{(2e^t)^{s-\frac{n-1}{2}}e^{i\pi s}\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\times\right.\\
&\qquad\qquad\left.\times{}_2F_1\left(s+\frac{n-1}{2},-s+\frac{n-1}{2};\frac{n}{2};\frac{1+\tilde{Z}(t,\vec{\zeta};t',\xi')}{2}\right)\right\}.
\end{aligned}\tag{B.6}$$

From the asymptotic behavior of hypergeometric function:

$$\begin{aligned}
&{}_2F_1\left(s+\frac{n-1}{2},-s+\frac{n-1}{2};\frac{n}{2};\frac{1+x}{2}\right)\\
&\underset{|x|\to\infty}{\approx}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(2s)}{\Gamma\left(s+\frac{n-1}{2}\right)\Gamma\left(s+\frac{1}{2}\right)}\left(-\frac{x}{2}\right)^{s-\frac{n-1}{2}}+\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(-2s)}{\Gamma\left(-s+\frac{n-1}{2}\right)\Gamma\left(-s+\frac{1}{2}\right)}\left(-\frac{x}{2}\right)^{-s-\frac{n-1}{2}}.
\end{aligned}\tag{B.7}$$

we get the integral as

$$e^{i\pi s}\frac{2^s\,(2\pi)^{\frac{n-1}{2}}\,\Gamma(s)}{\Gamma\left(s+\frac{n-1}{2}\right)}\left(\cosh t'\,\vec{\zeta}\cdot\vec{\xi}'+\sinh t'-i\epsilon\right)^{s-\frac{n-1}{2}},\tag{B.8}$$

Finally, the integral with respect to $\vec{\zeta}$ gives

$$\begin{aligned}
F_n(t,\vec{\xi};t',\vec{\xi}')&=Q_0\,e^{i\pi s}\int_{S^{n-1}} d^{n-1}\Omega(\vec{\zeta})\left(\cosh t\,\vec{\zeta}\cdot\vec{\xi}+\sinh t+i\epsilon\right)^{-s-\frac{n-1}{2}}\times\\
&\qquad\times\left(\cosh t'\,\vec{\zeta}\cdot\vec{\xi}'+\sinh t'-i\epsilon\right)^{s-\frac{n-1}{2}}\\
&=Q_0\,V_{n-1}\,{}_2F_1\left(s+\frac{n-1}{2},-s+\frac{n-1}{2};\frac{n}{2};\frac{1+\tilde{Z}}{2}\right).
\end{aligned}\tag{B.9}$$

C. Explicit expression of the two-point function for vertex operators

The two-point function of vertex operators, $(\Omega|\,V^\dagger(x)\,V(x')\,\Omega)\equiv\exp\left(\frac{1}{L^2}\,\mathcal{G}_n(x;x')\right)$ was given by eq. (6.65) and eq. (6.66). An explicit expression for G_n can be deduced from the differential equation (6.68) and the boundary conditions at $Z=\pm1$ which gives for n even:

$$\begin{aligned}
G_n(x;x')&=-\log\left\{2\left(1-\tilde{Z}(x;x')\right)\right\}+\sum_{m=1}^{\frac{n-2}{2}}\frac{\left(\frac{2-n}{2}\right)_m}{m\,(2-n)_m}\left\{\frac{2^m}{\left(1-\tilde{Z}(x;x')\right)^m}-1\right\}+\\
&\quad+\psi\left(\frac{n-1}{2}\right)-\psi\left(\frac{n}{2}\right)+\psi\,(1)-\psi\left(\frac{1}{2}\right),
\end{aligned}\tag{C.1}$$

and for n odd:

$$G_n(x;x') = \frac{n-1}{n}\left(1-\tilde{Z}(x;x')\right){}_3F_2\left(1\,,\,1\,,\,n\,;\,2\,,\,1+\frac{n}{2}\,;\,\frac{1-\tilde{Z}(x;x')}{2}\right)+ \qquad (C.2)$$
$$+\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}{2^{n+1}\,\Gamma(n-1)}\,\tilde{Z}(x;x')\,{}_2F_1\left(\frac{1}{2}\,,\,\frac{n}{2}\,;\,\frac{3}{2}\,;\,\left(\tilde{Z}(x;x')\right)^2\right)+\psi\left(\frac{n-1}{2}\right)-\psi\left(\frac{n}{2}\right).$$

These two expressions have a logarithmic behavior in the IR at large $|Z|$.

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